Online Planning in MDPs: Monte Carlo Methods

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Recap & Preview

- Last time: started *online planning* for MDPs
  - Current state known
  - Agent “in the wild”
  - Interleaving planning and execution

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- Considered *reachability* and *heuristics*
  - Expectimax search exploits reachability
  - Leaf heuristic evaluation, RTDP, determinization use heuristics
- Some MDPs are still too hard!
  - One hard case: *very large transition distributions*
  - Another hard case: *long horizons and sparse rewards*

MDPs with Very Large Transition Distributions

Recall Bellman backups:

- Given state s, for each a, *for each possible next state* s', update $V(s)$.

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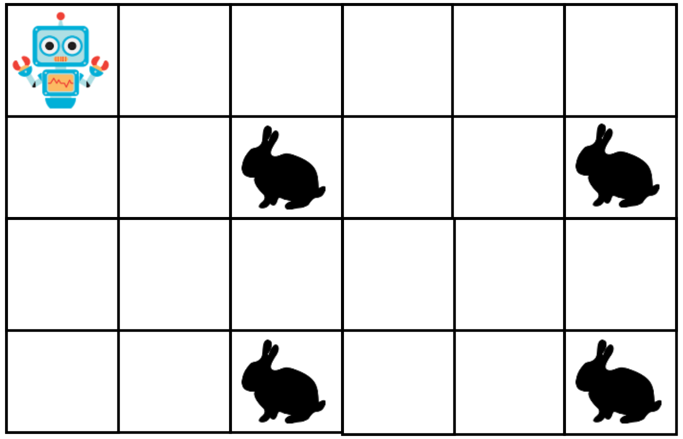
2. Server farm; any server might fail with small probability
3. Pathological MDP, small probability of transitioning anywhere



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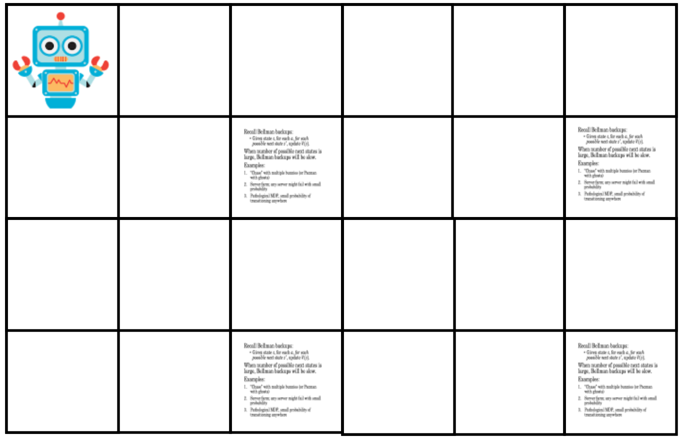
Almost all methods we’ve seen use Bellman backups. What’s the exception?



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We're going to need some new planning algorithms…

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MonteCarloBellmanBackup$(s, \mathtt{V}, \mathcal{S}, \mathcal{A}, P, R, \gamma, w)$

$\mathtt{vs} = -\infty$ // New estimate for $V(s)$
for each $\mathtt{a} \in \mathcal{A}$
    $\mathtt{qsa} = 0$ // New estimate for $Q(s, a)$
    repeat w times
        $\mathtt{ns} \sim P(\cdot \mid \mathtt{s}, \mathtt{a})$ // Simulator access only
        $\mathtt{qsa} = \mathtt{qsa} + \frac{1}{w}(R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \mathtt{V}[\mathtt{ns}])$
    $\mathtt{vs} = \max(\mathtt{vs}, \mathtt{qsa})$
return $\mathtt{vs}$

Finite horizon case is analogous

Sparse Sampling

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- Nice property: can get optimality guarantees that depend only on w and H, *not* on $|\mathcal{S}|$ (Kearns, Mansour, and Ng 1999).

SPARSESAMPLING$(s_0, \mathcal{S}, \mathcal{A}, P, R, H, w)$

// a.k.a. MONTECARLOEXPECTIMAXSEARCH
return $\operatorname{argmax}_{\mathsf{a}} \mathrm{Q}(s_0, \mathsf{a}, 0, \mathcal{S}, \mathcal{A}, P, R, H, w)$

$\mathrm{Q}(\mathsf{s}, \mathsf{a}, \mathsf{t}, \mathcal{S}, \mathcal{A}, P, R, H, w)$

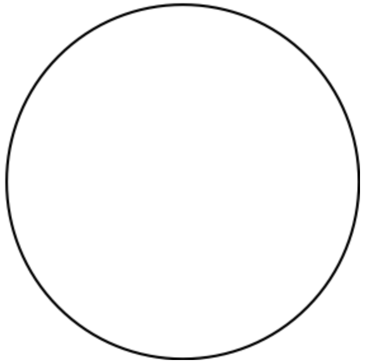
    qsa $= 0$
    repeat w **times**
        ns $\sim P(\cdot \mid \mathsf{s}, \mathsf{a})$ // Simulator access only
        $\mathsf{v}_{ns} = \mathrm{V}(\mathsf{s}, \mathsf{t}, \mathcal{S}, \mathcal{A}, P, R, H, w)$
        qsa $=$ qsa $+ \frac{1}{w}(R(\mathsf{s}, \mathsf{a}, \mathsf{ns}) + \gamma \mathsf{v}_{ns})$
    return qsa

$\mathrm{V}(\mathsf{s}, \mathsf{t}, \mathcal{S}, \mathcal{A}, P, R, H, w)$

if $t = H$
    return 0
return $\max_{\mathsf{a}} \mathrm{Q}(\mathsf{s}, \mathsf{a}, \mathsf{t}, \mathcal{S}, \mathcal{A}, P, R, H, w)$

Sparse sampling

$H = 2, w = 3$



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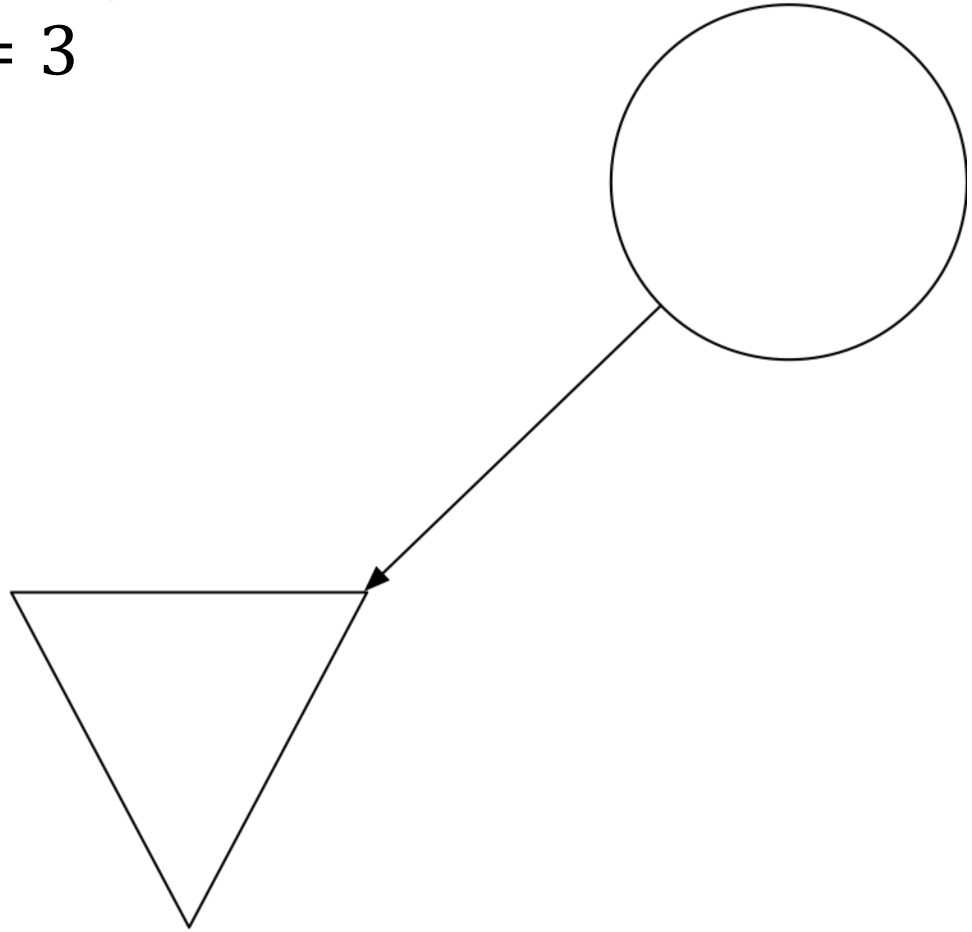

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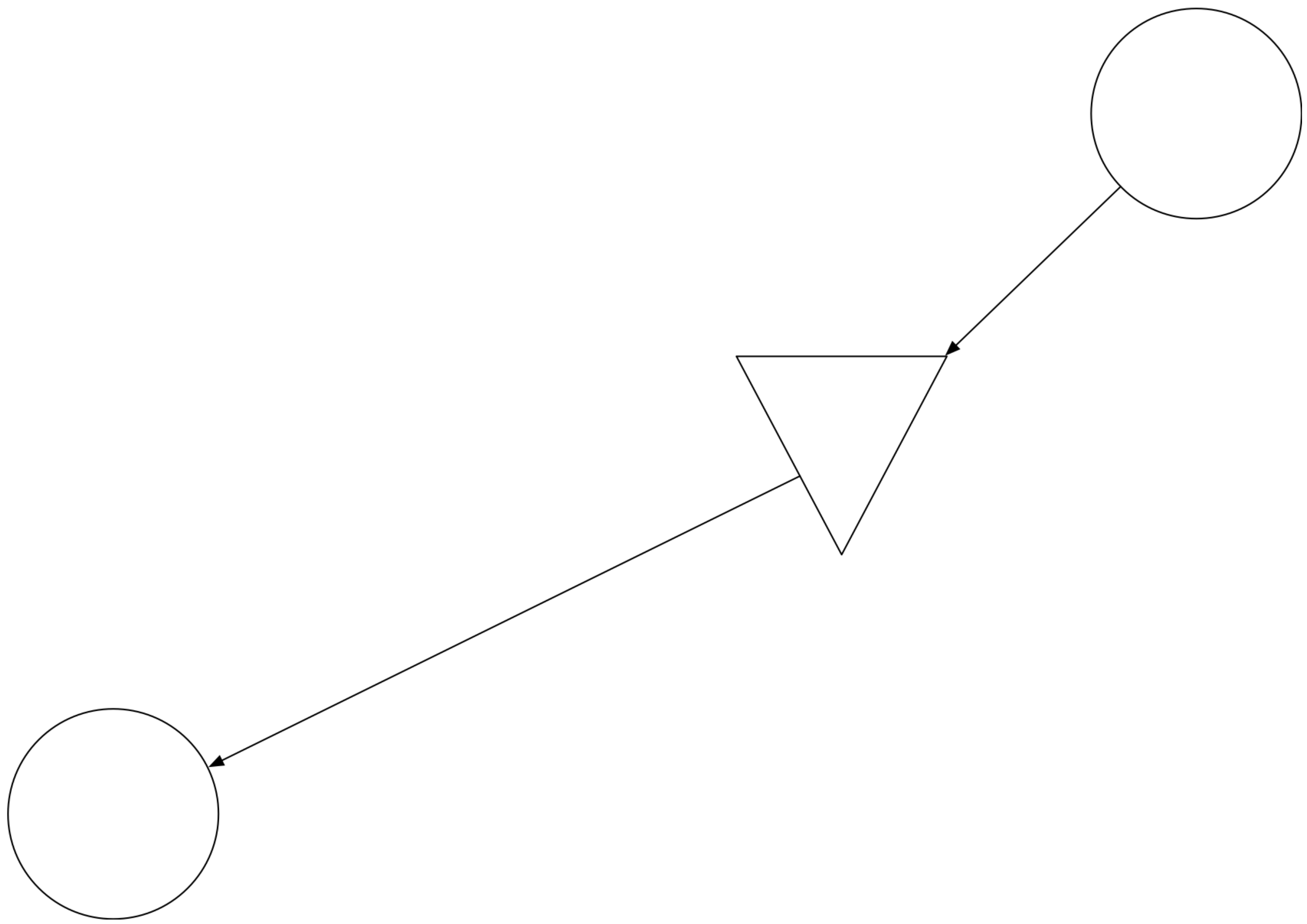

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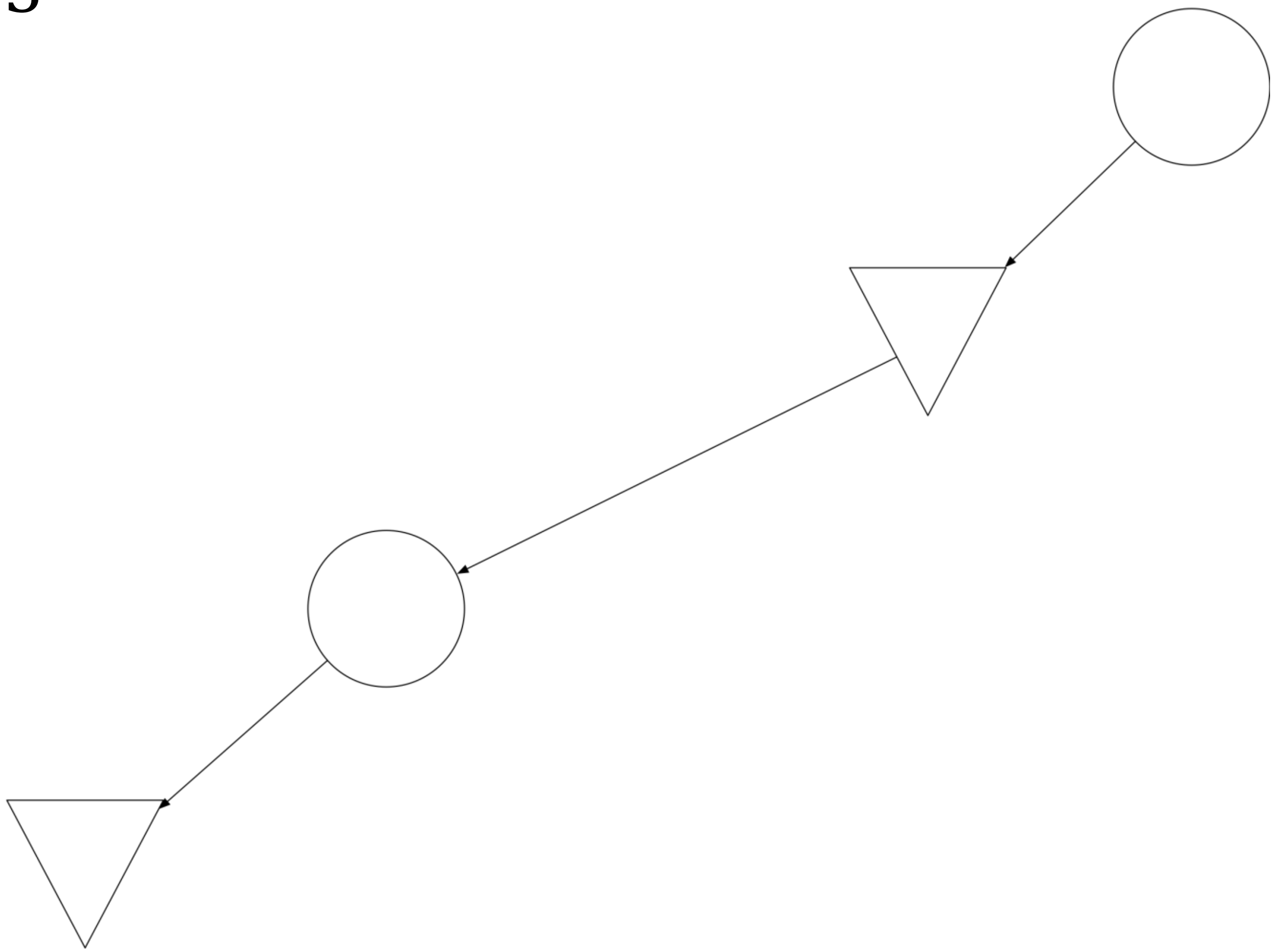

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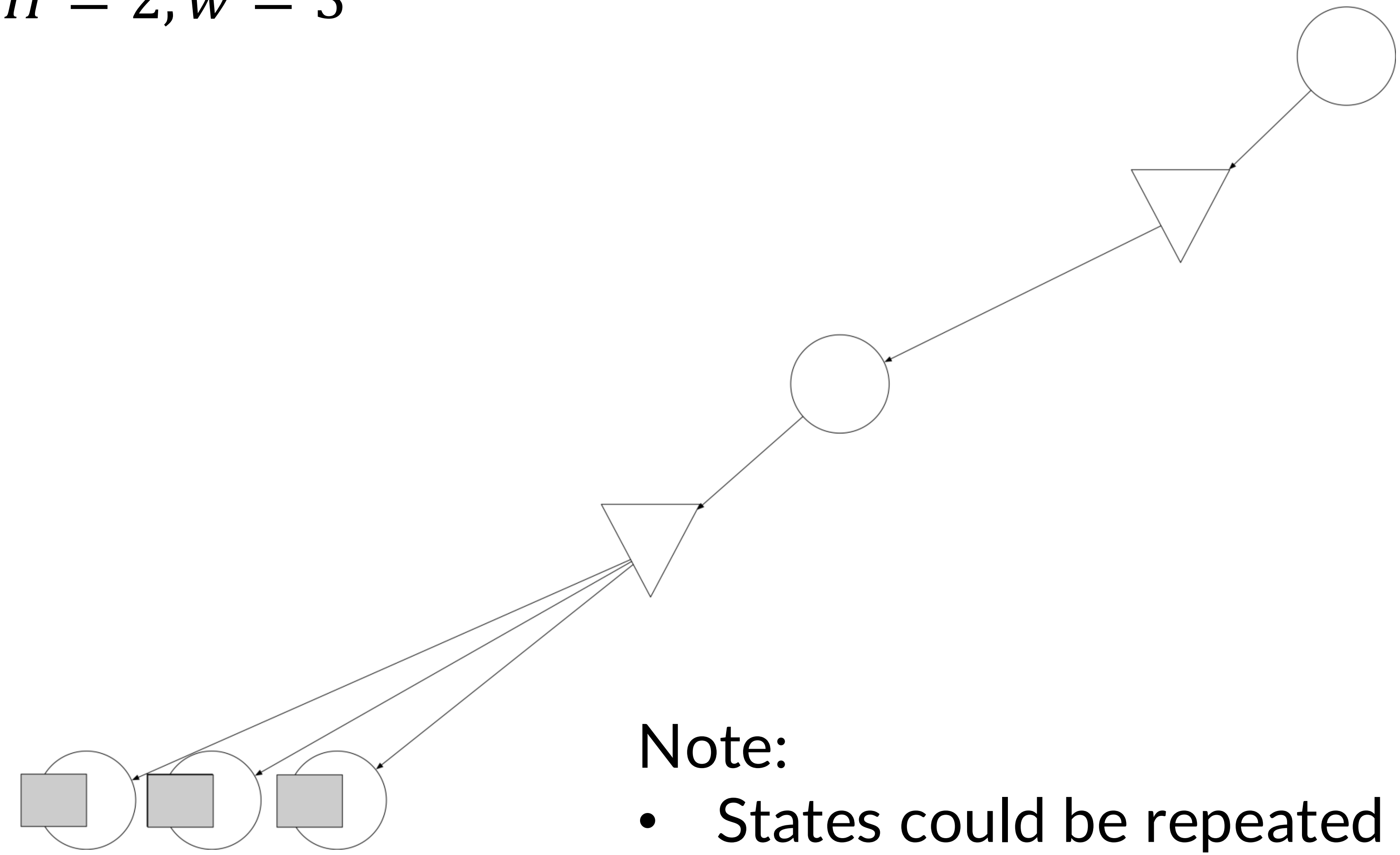

Note:
- States could be repeated
- Actual # successors could be >> 3

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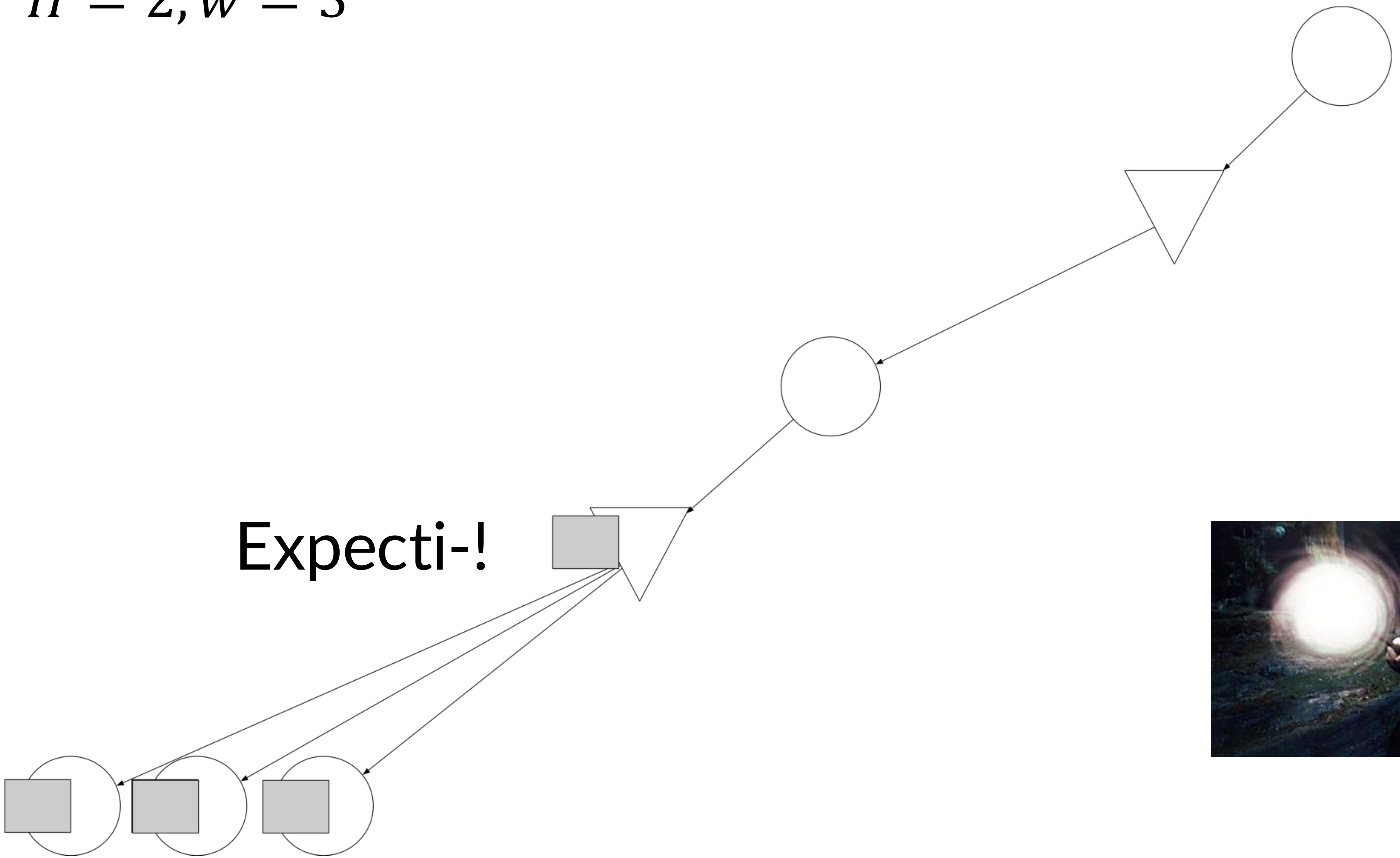

Expecti-!

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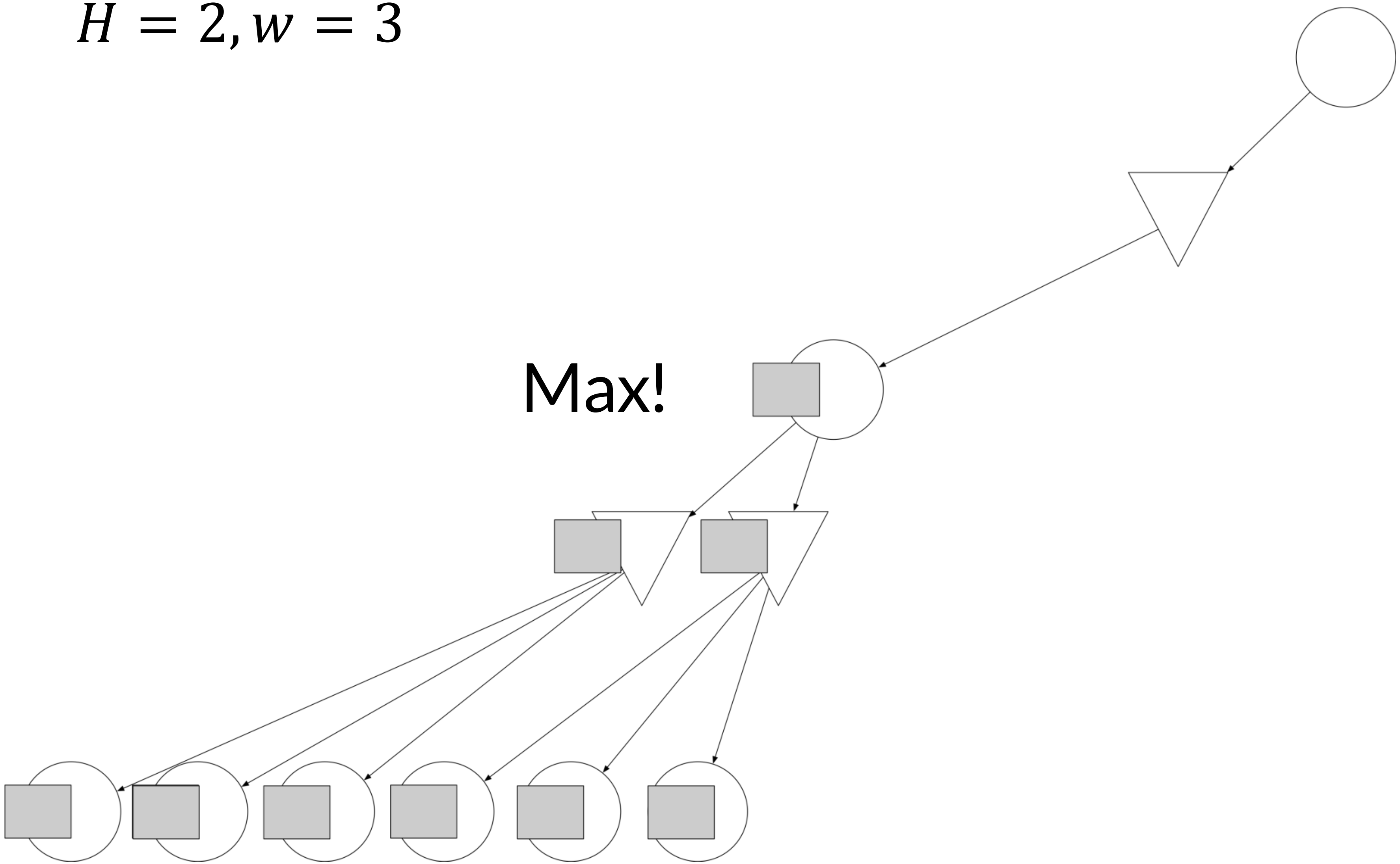

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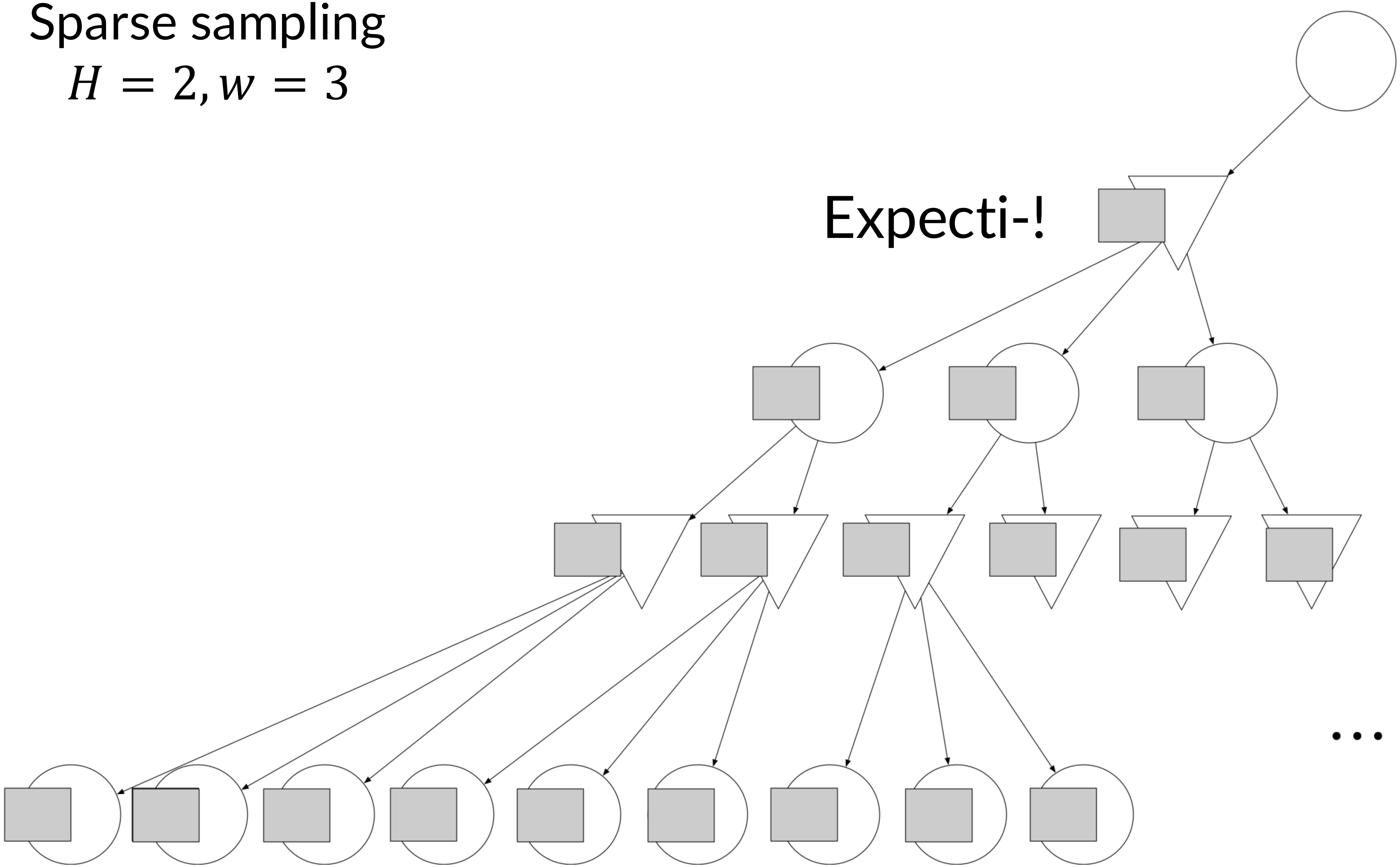

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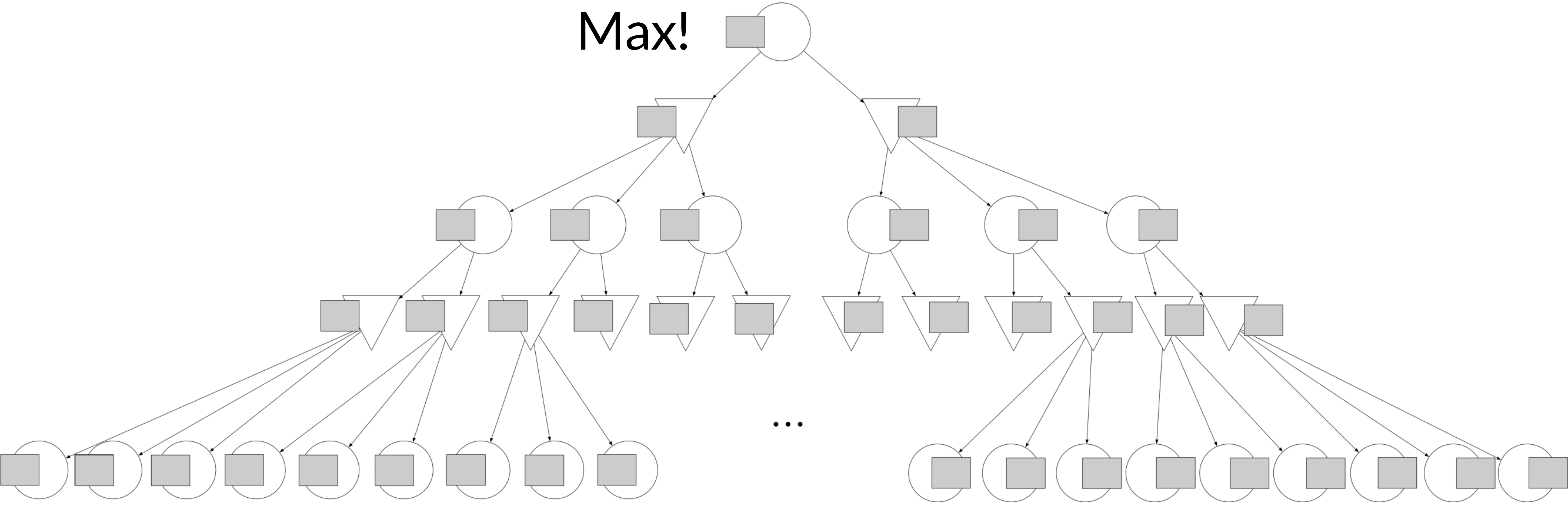

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- Recall limitation of expectimax: exhaustive AODAG building
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Let's study this in special case: $H = 1$.

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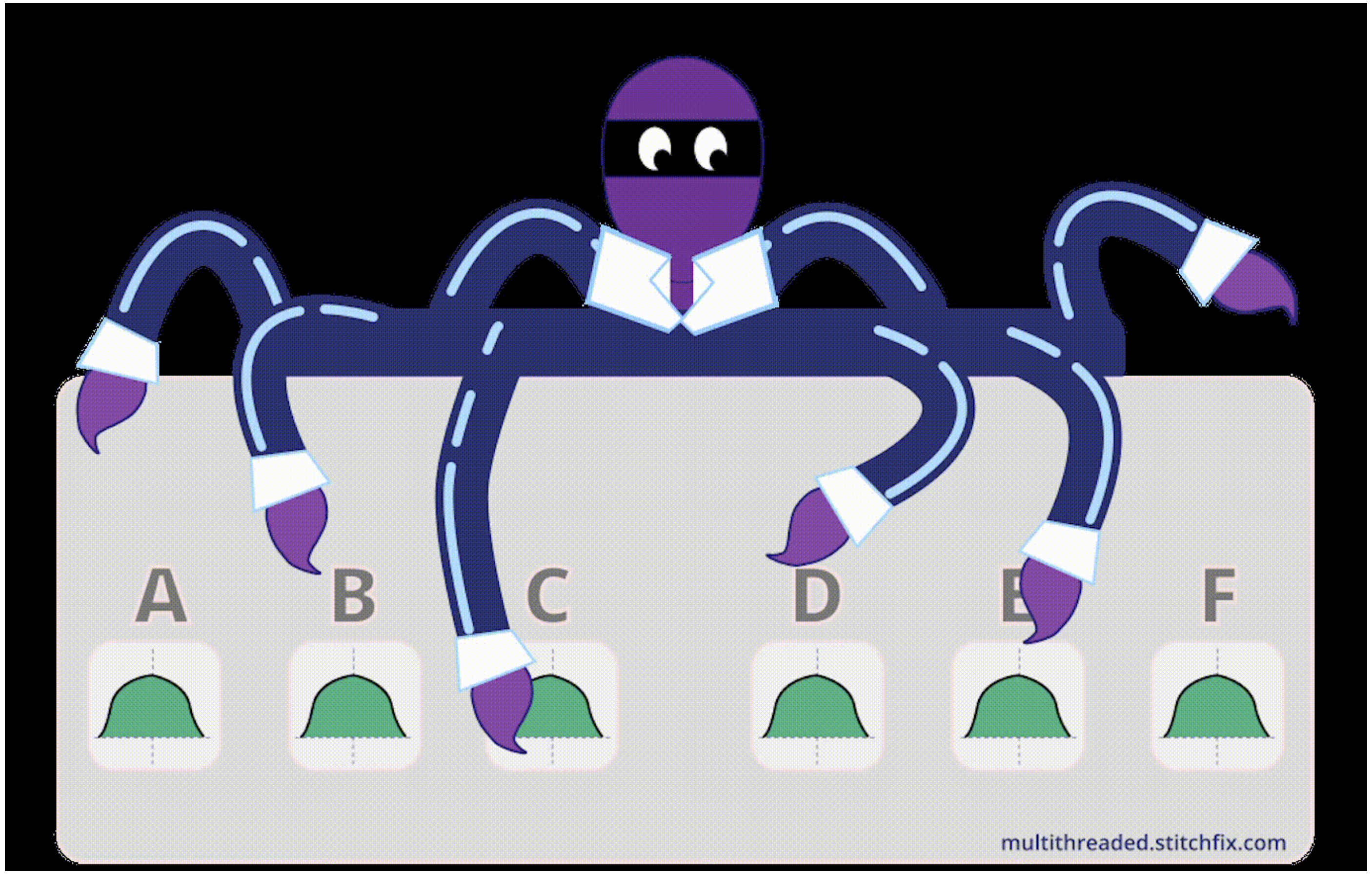

https://multithreaded.stitchfix.com/blog/2020/08/05/bandits/

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Consider finite horizon MDP, $H = 1$. Simulator access only.

And just one fixed initial state.

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What's the objective?

MAB: A Tale of Two Settings

Objective: minimize *regret*

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- After M samples, take one final action and receive r_{M+1}.
- Simple regret: $r^*_{M+1} - r_{M+1}$ where r^*_{M+1} is best possible under clairvoyant policy.

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When would each make more sense?

Strategies for MAB

- Most strategies maintain sample estimate of Q function:

$$\hat{Q}(s, a) = \frac{1}{|\mathcal{I}_a|} \sum_{i \in \mathcal{I}_a} r_i$$

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Why might "always select $\mathrm{argmax}_\mathrm{a}\ \hat{Q}(s, a)$" be a suboptimal strategy?

Strategies for MAB: ϵ-greedy

Epsilon-greedy strategy

- With probability ϵ, select random action (explore)
- Otherwise, select $\mathrm{argmax_a}\ \hat{Q}(s, a)$ (exploit)

If there's an action that has never been tried ($N(s\ , a) = 0$), select it.

Strategies for MAB: UCB

Upper confidence bounds (UCB)

Main idea: *optimism in the face of uncertainty*.

- New restaurant in town! I don’t know if it’s good, but optimistically, it might be fantastic! Let’s eat.
- New course offering! I don’t know if it’s good, but optimistically, it might be. Let’s take it!

Why is optimism in the face of uncertainty a good principle?

- If your optimistic predictions are correct, you'll be thrilled!
- If they're not, you will quickly discover that you were wrong from the new data.
- Contrast with pessimism.

Being Optimistic with Confidence Bounds

- Suppose I believe that with 95% probability, $\hat{Q}(s, a_1)$ is between -1.25 and 4.75.

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- I also think that with 95% probability, $-3.0 \ \leq \hat{Q}(s, a_2) \leq 5.0$.

- Optimistically, $\hat{Q}(s, a_2) > \hat{Q}(s, a_1)$. So, I'll choose a_2!

Recipe for Deriving UCB Algorithms

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- For each action $a \in \mathcal{A}$, define random variables $X_a^1, \dots X_a^n$ where X_a^i represents the reward for the i^{th} try of action a.

- Note that these X_a^i are i.i.d. with distribution $R(s, a, S')$, where $S' \sim P(s' \mid s, a)$, which has mean $Q(s, a)$.

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- Let $\hat{X}_a^n = \frac{1}{n} \sum_{i=1}^{n} X_a^i$. Represents $\hat{Q}(s, a)$ after n tries of a.

- Make some assumptions about the distribution (e.g., it is subgaussian) and use some concentration bounds (e.g, Chebyshev) to derive an inequality like...

Recipe for Deriving UCB Algorithms

$$P\left(Q(s,a) \geq \hat{X}_a^n + \sqrt{\frac{2\log\left(\frac{1}{\delta}\right)}{n}}\right) \leq \delta \qquad \text{For any } \delta \in (0,1)$$

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Given a desired confidence level, like $(1-\delta)$ = 0.95, the most optimistic plausible estimate of the true value is $\hat{X}_a^n + constant$.

Recipe for Deriving UCB Algorithms

As n gets larger, or as $1 - \delta$ gets larger, bound gets tighter.

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Upper confidence bounds (UCB)

Note resemblance to concentration bounds!

Idea: construct confidence intervals for $\hat{Q}$, then be *optimistic in the face of uncertainty.*

$$\text{At step } m\text{, select: } \operatorname{argmax}_{\mathrm{a}} \left[\hat{Q}(s,a) + \frac{\phi(m)}{\sqrt{N(s,a)}}\right]$$

where ϕ can be several functions; often $\phi(m) = c\sqrt{\log(m)}$ for a hyperparameter c.

Intuition: as number of tries increases, shift from exploration to exploitation.

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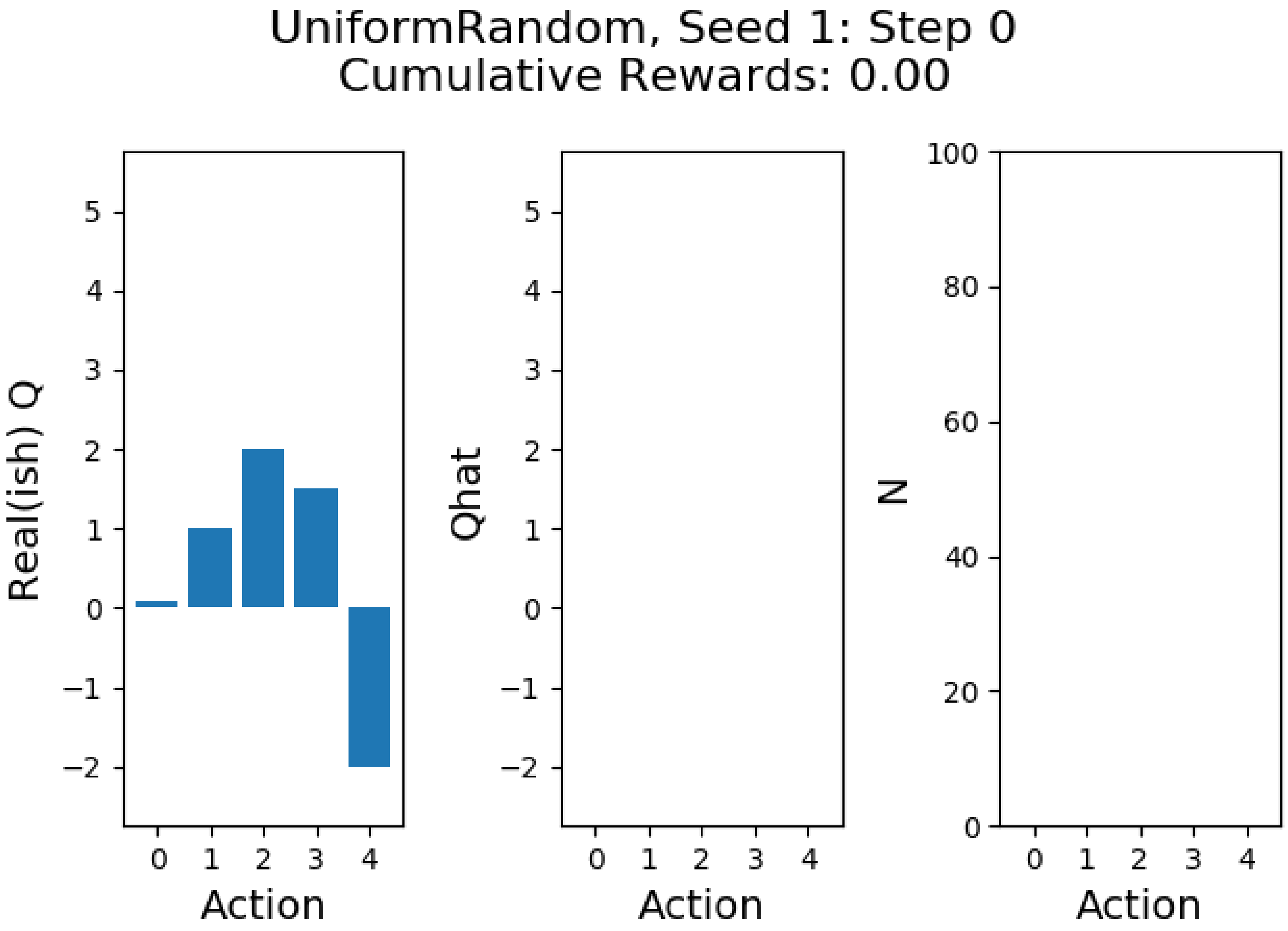
Upper confidence bounds (UCB)

- UCB attains optimal cumulative regret (Lai & Robbins 1985)
- It does not attain optimal simple regret (Bubeck et al. 2010)
- But it's widely used in planning contexts nonetheless, and works well in practice



UniformRandom, Seed 1: Step 0
Cumulative Rewards: 0.00
Real(ish) Q
Qhat
N
Action
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Action
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2
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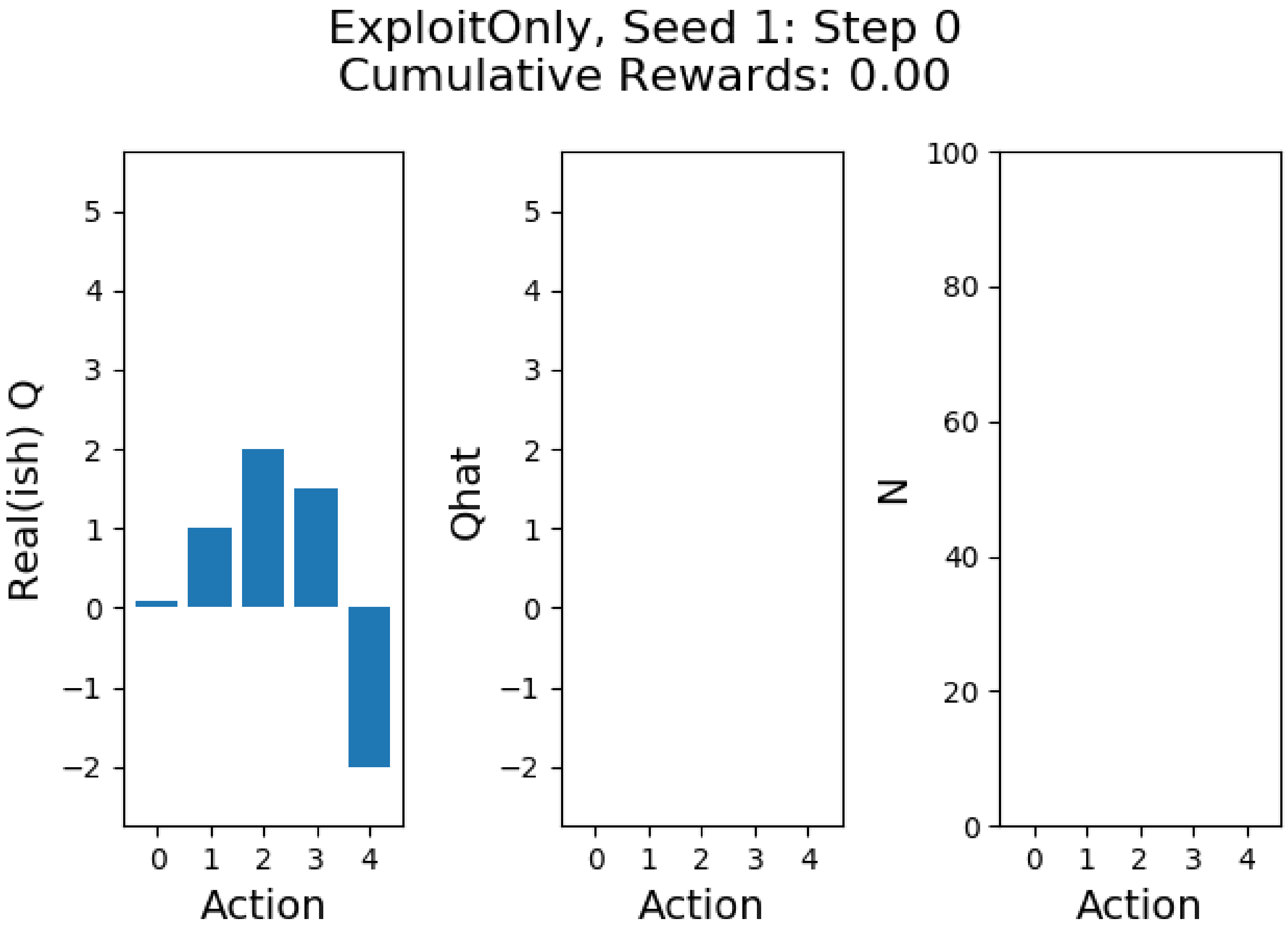

ExploitOnly, Seed 1: Step 0
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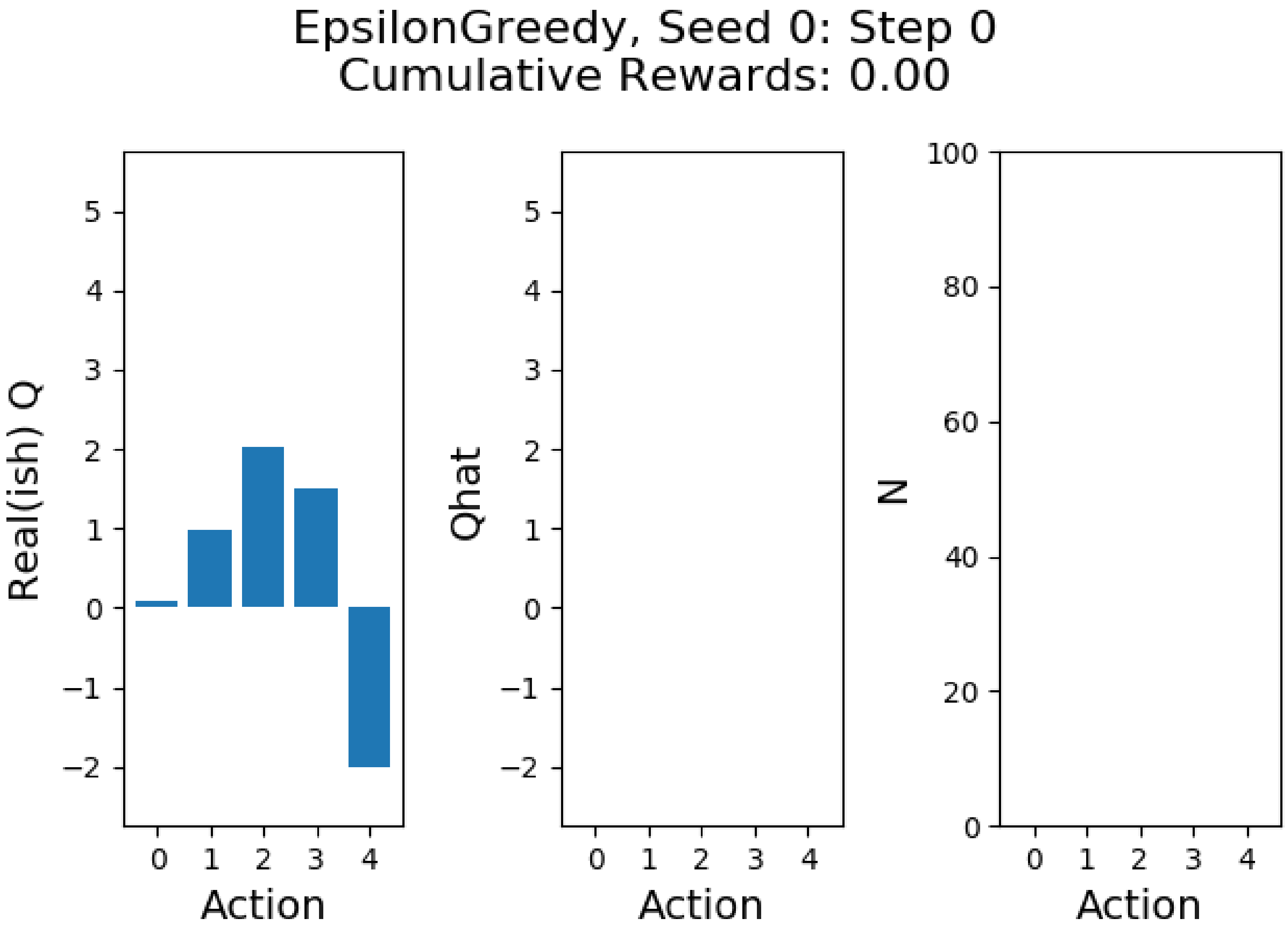
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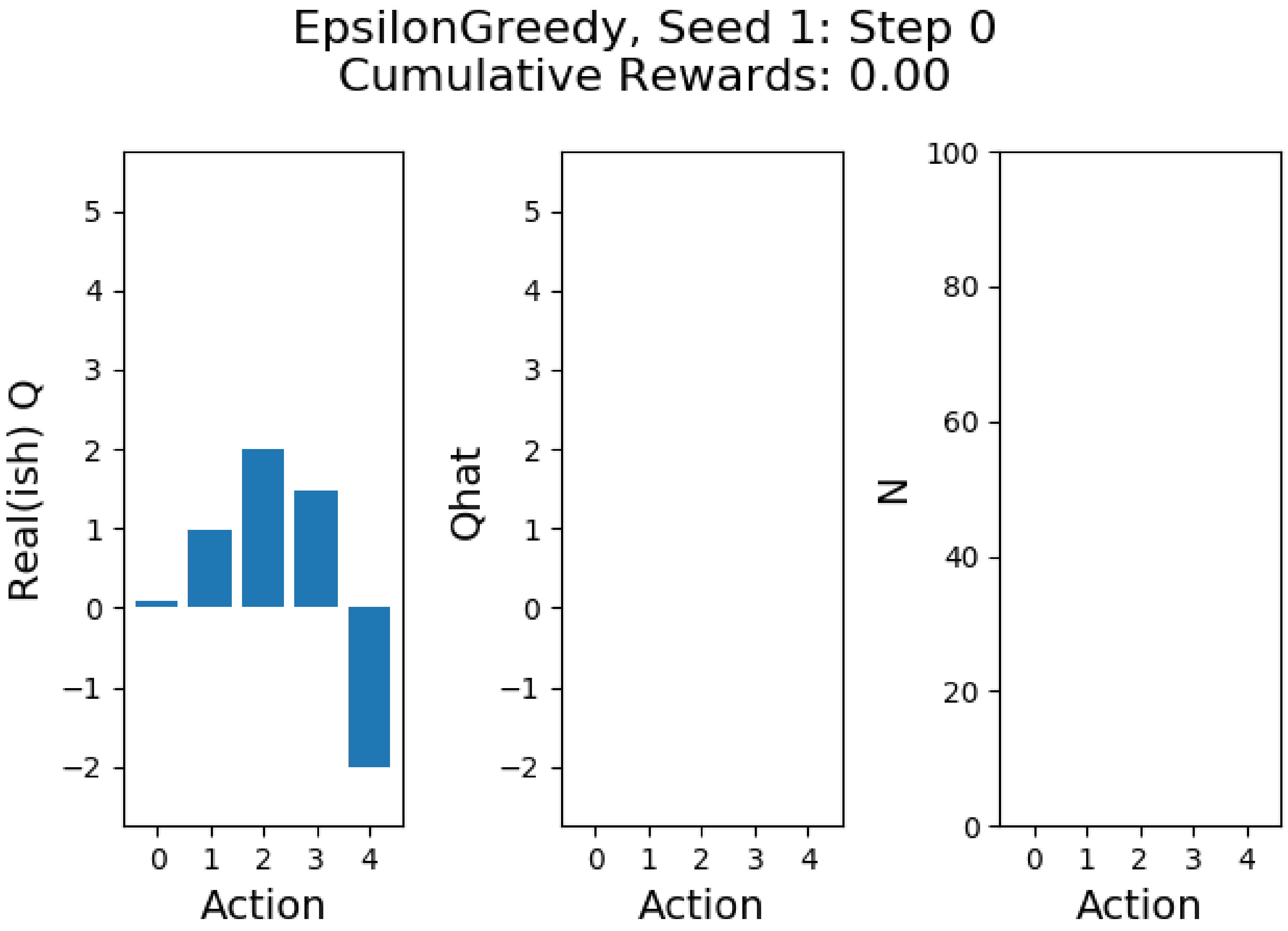
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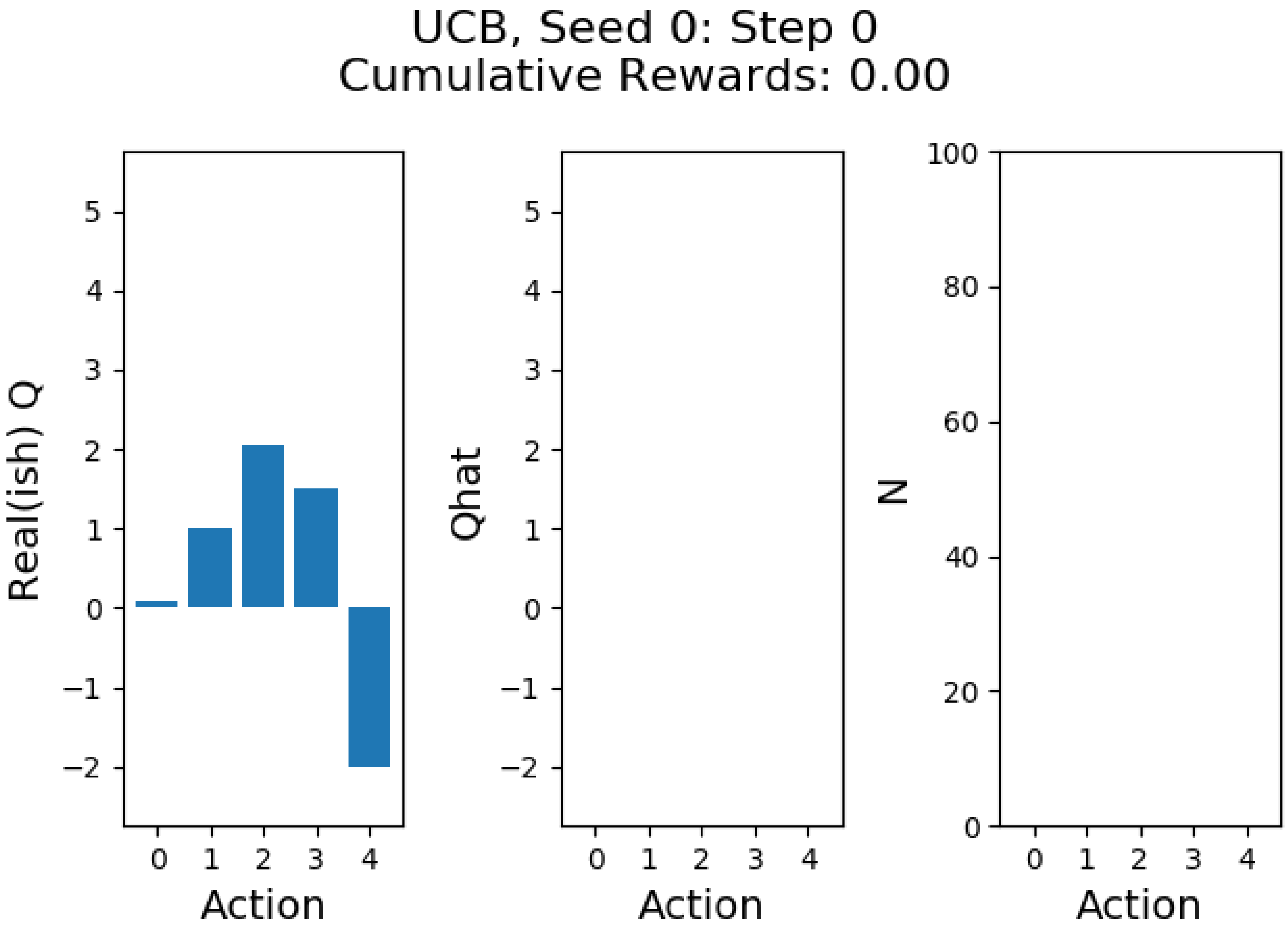
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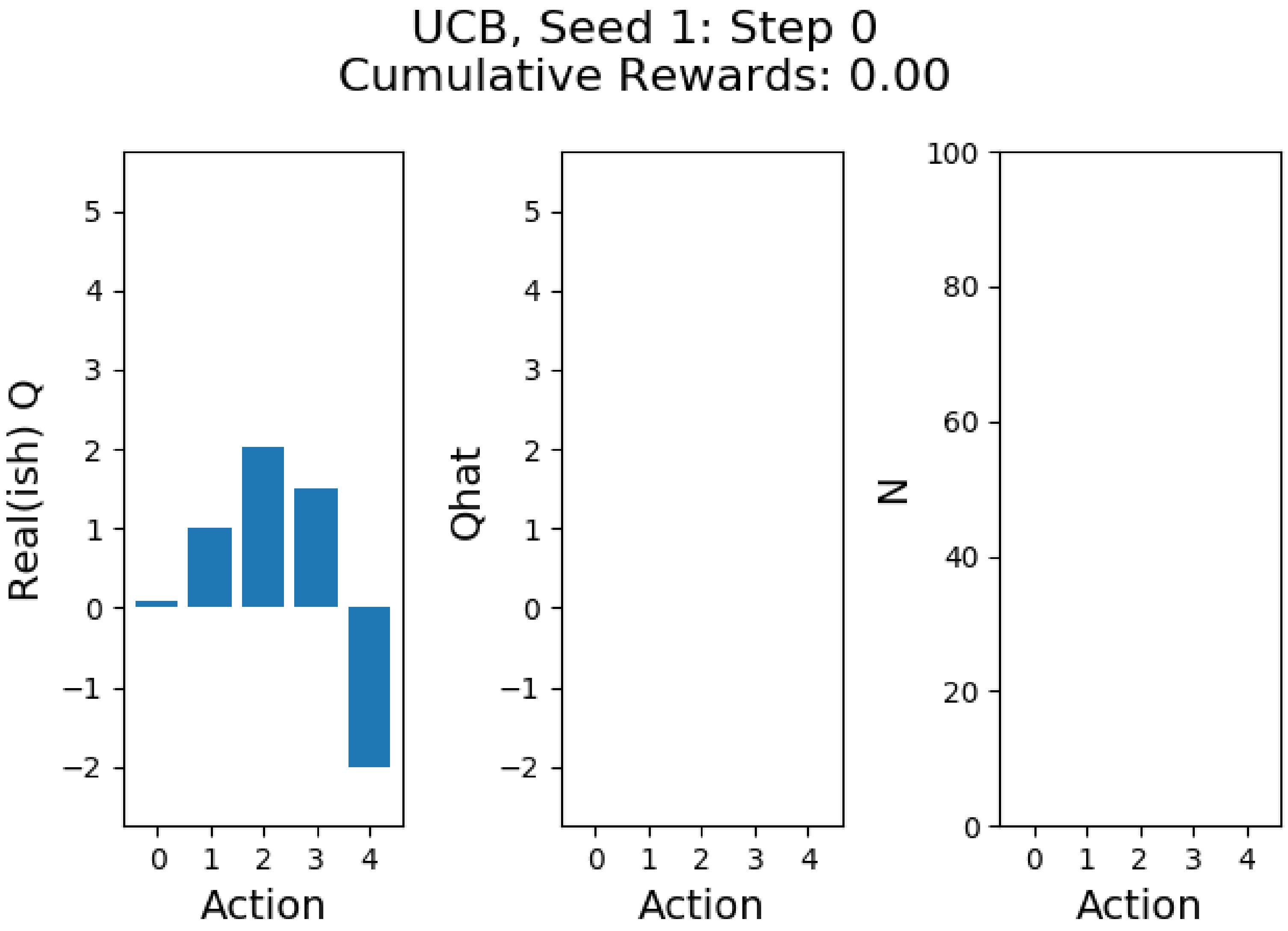
UCB, Seed 1: Step 0
Cumulative Rewards: 0.00
Real(ish) Q
Qhat
N
Action
Action
Action
5
4
3
2
1
0
−1
−2
100
80
60
40
20
0
0 1 2 3 4

	Uniform Random	Exploit Only	Epsilon Greedy	UCB
Mean **Cumulative** Reward	55.70	141.98	134.75	149.70
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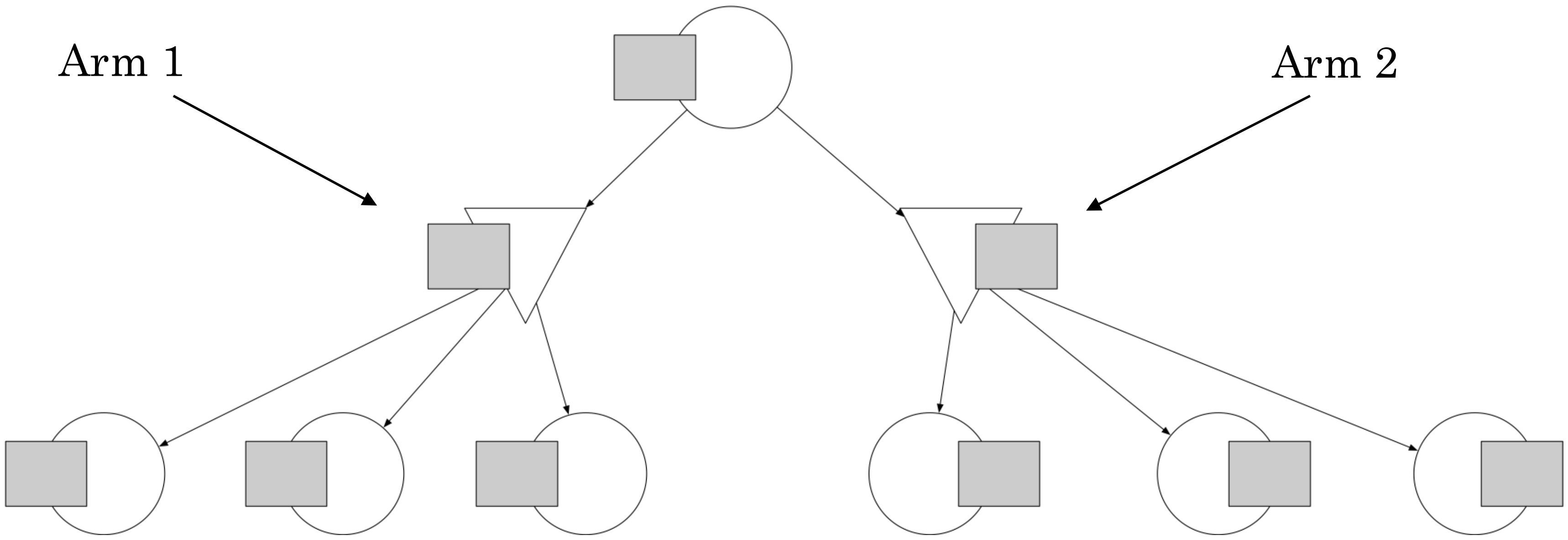
Lots more on Bandits:

- "Bandit Algorithms." Lattimore & Szepesvari (2020). Free online.
- https://banditalgs.com/

Recursive Bandits

- Sparse sampling with $H = 1 \approx$ "Uniform Random" for MAB.



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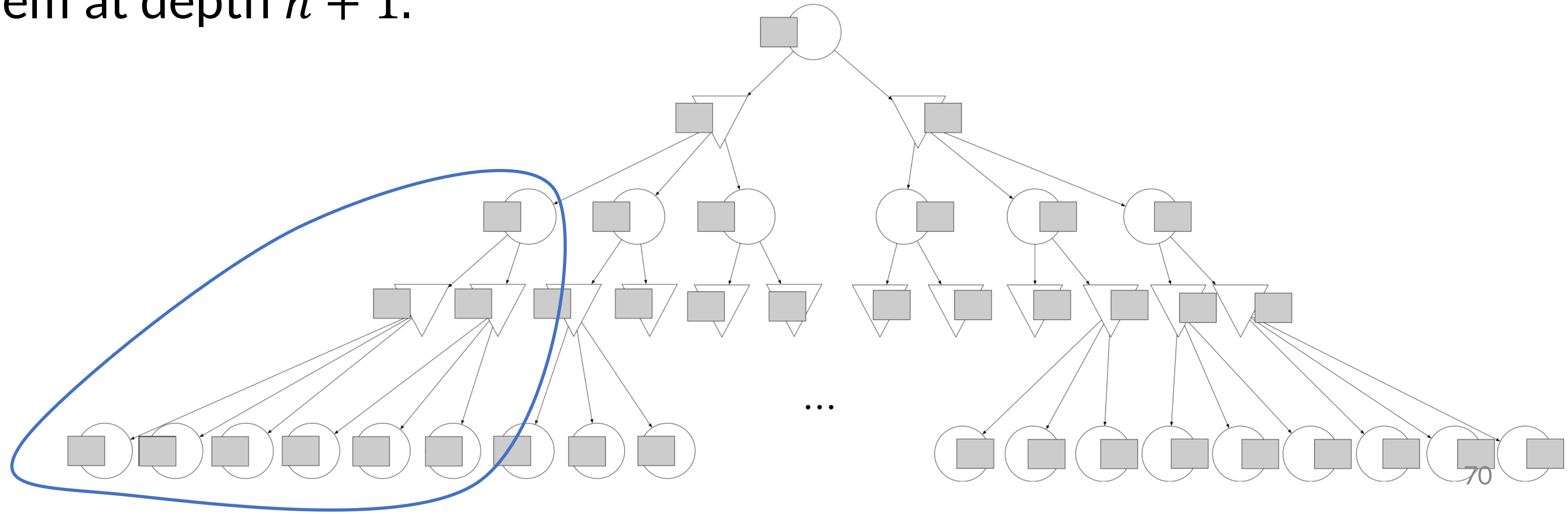

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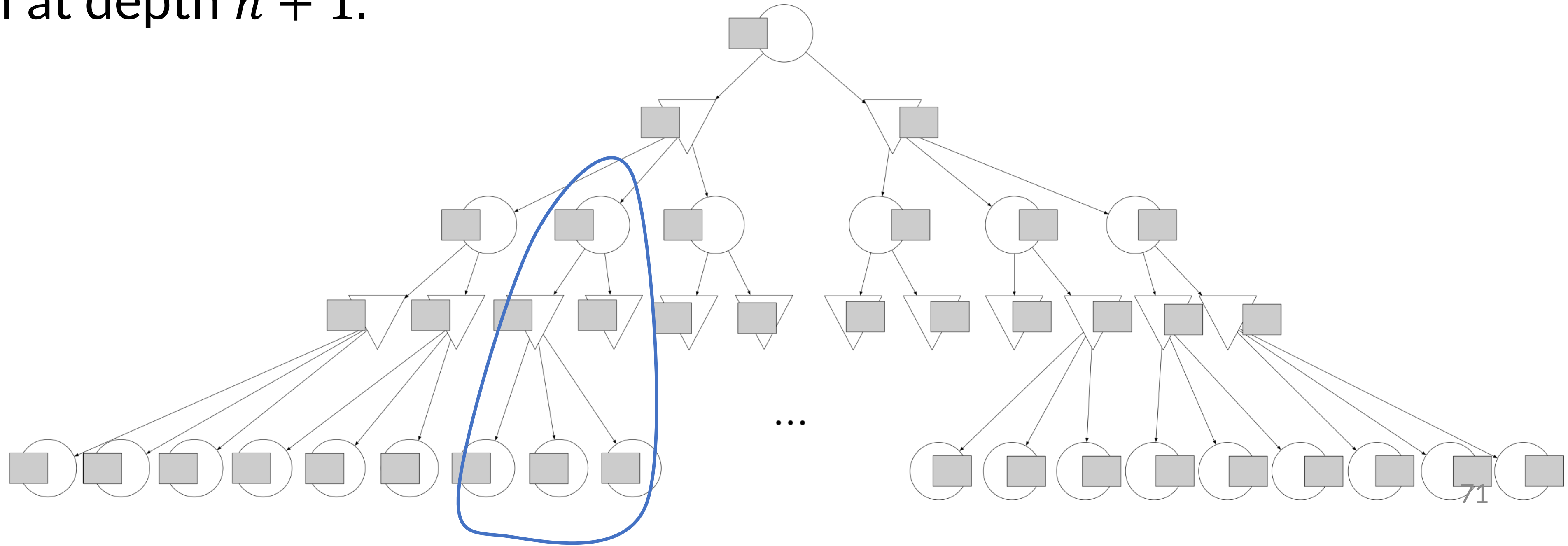

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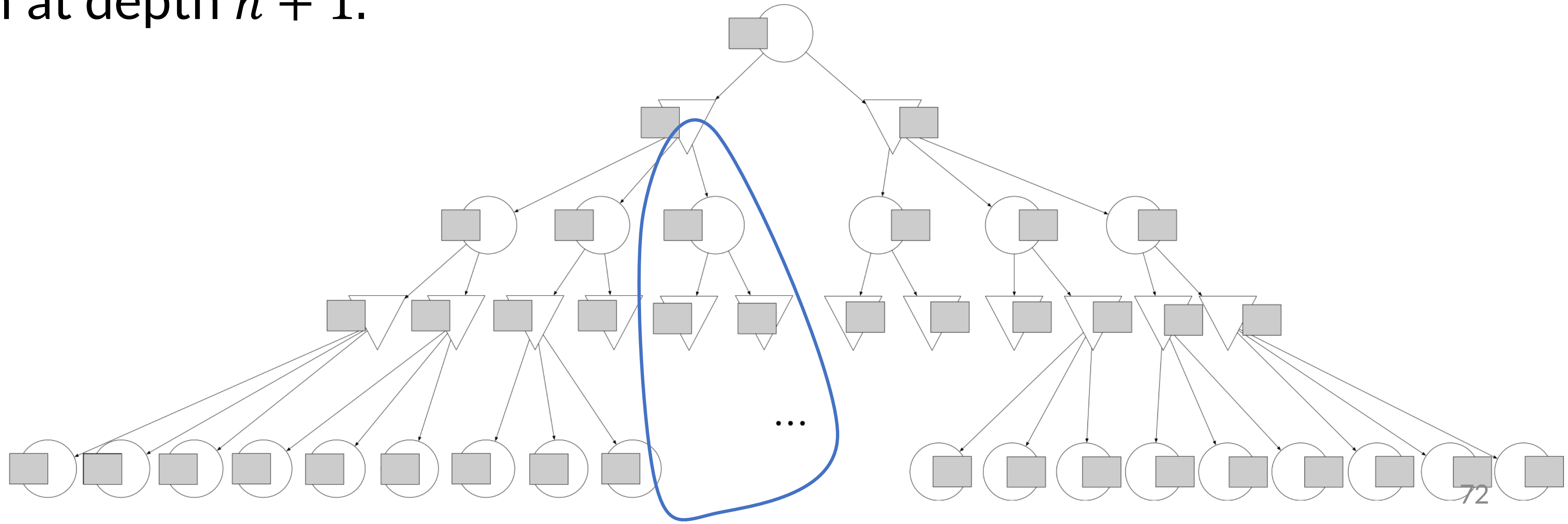

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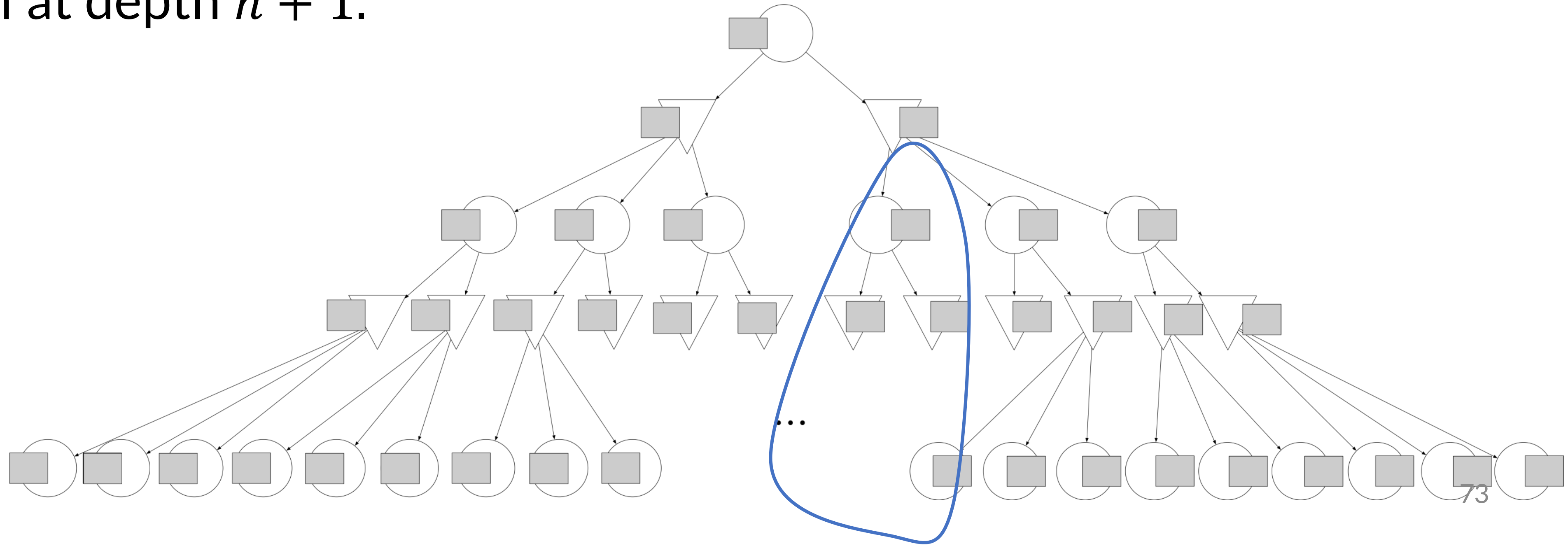

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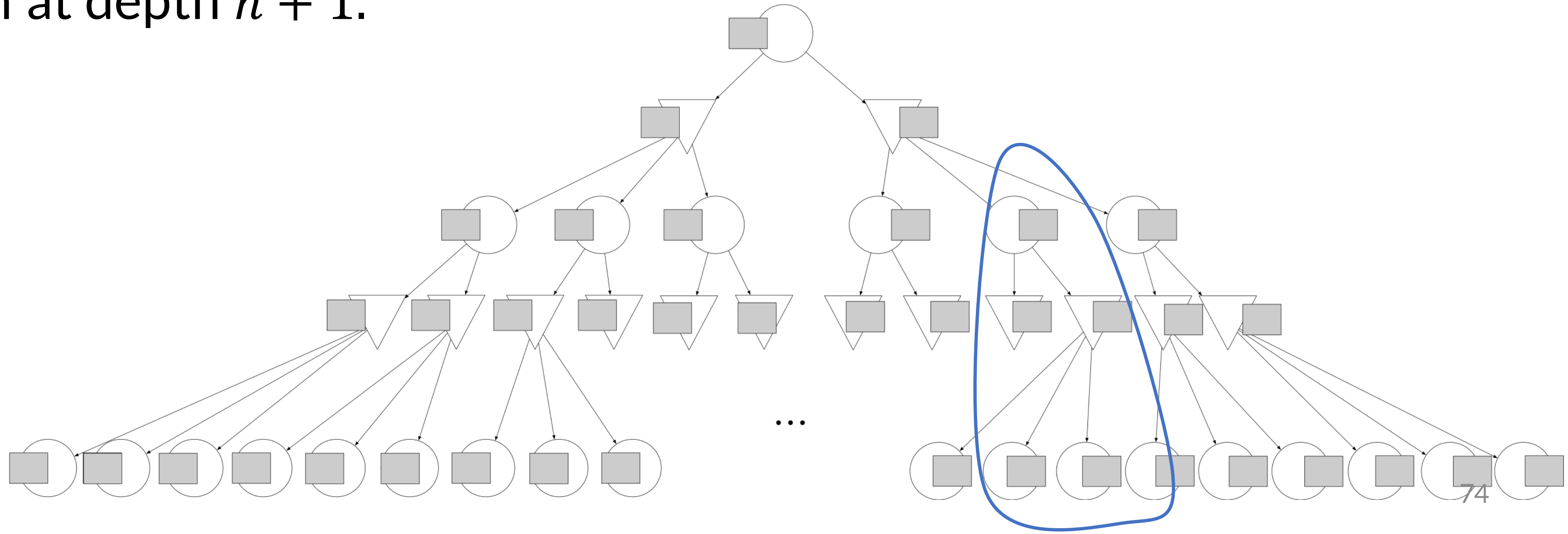

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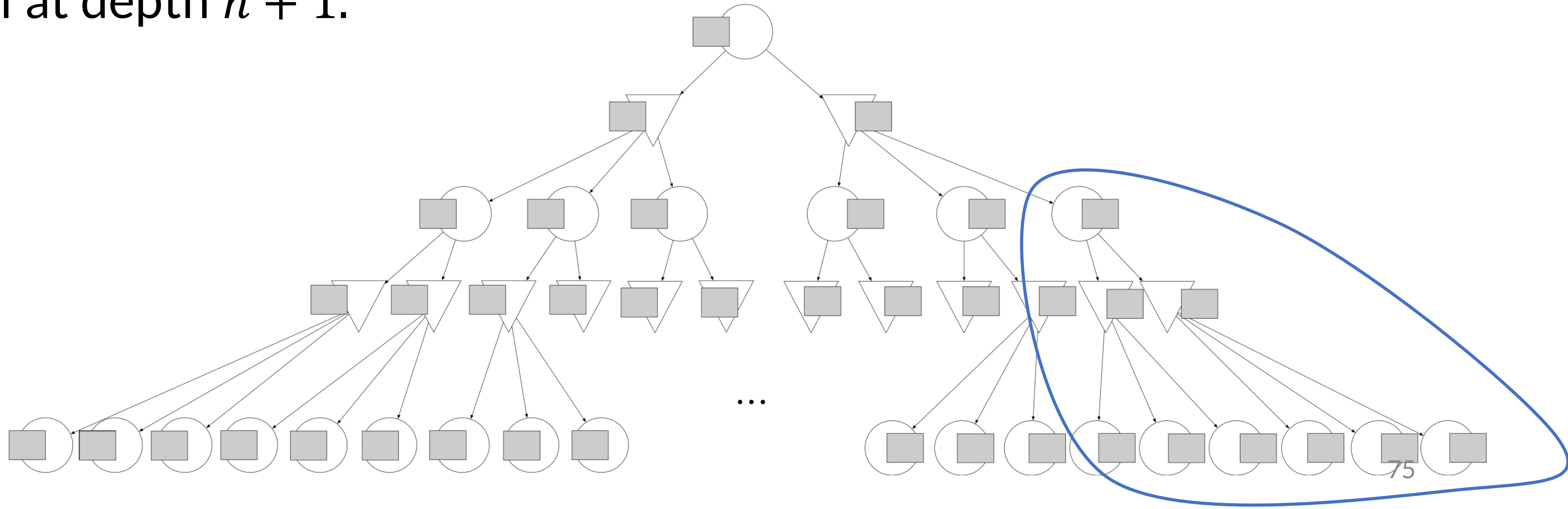

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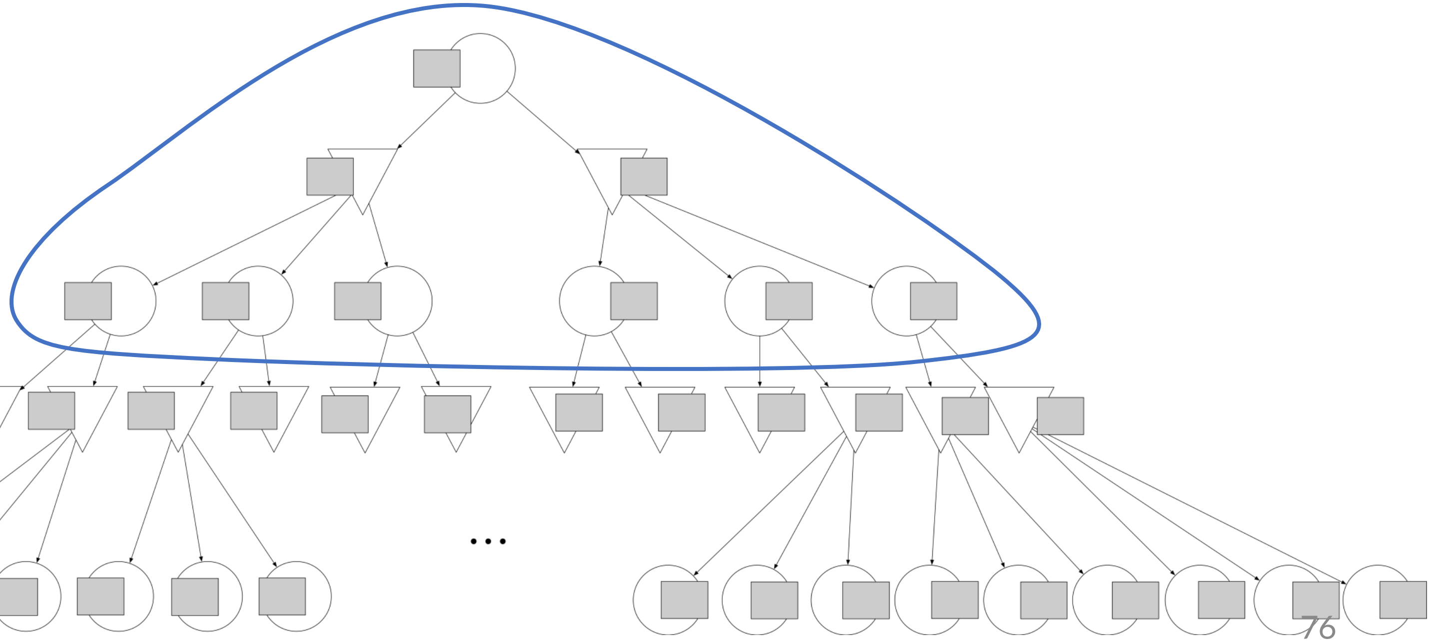

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- Could we recursively apply bandit approaches like UCB within sparse sampling?
  - Sure!

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- However, beyond the root, the story is less clear [1].
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- But if what I need is $V(s) = Q(s, a)$, then I need more a samples.
- For this reason, some works (e.g. [2]) advocate using one strategy at/near the root, and a different strategy elsewhere.

[1] “Simple Regret Optimization in Online Planning for Markov Decision Processes.” Feldman & Domshlak (2012).
[2] “Minimizing Simple and Cumulative Regret in Monte-Carlo Tree Search.” Pepels et al. (2014).

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  - Anytime performance: evaluation of the best policy found for any given computational budget (e.g., wall clock time)
- In general, sparse sampling completely evaluates each subtree before returning to the parent.

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One new idea: estimating heuristics with *rollouts*.

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When would rollouts give good or bad heuristic estimates?

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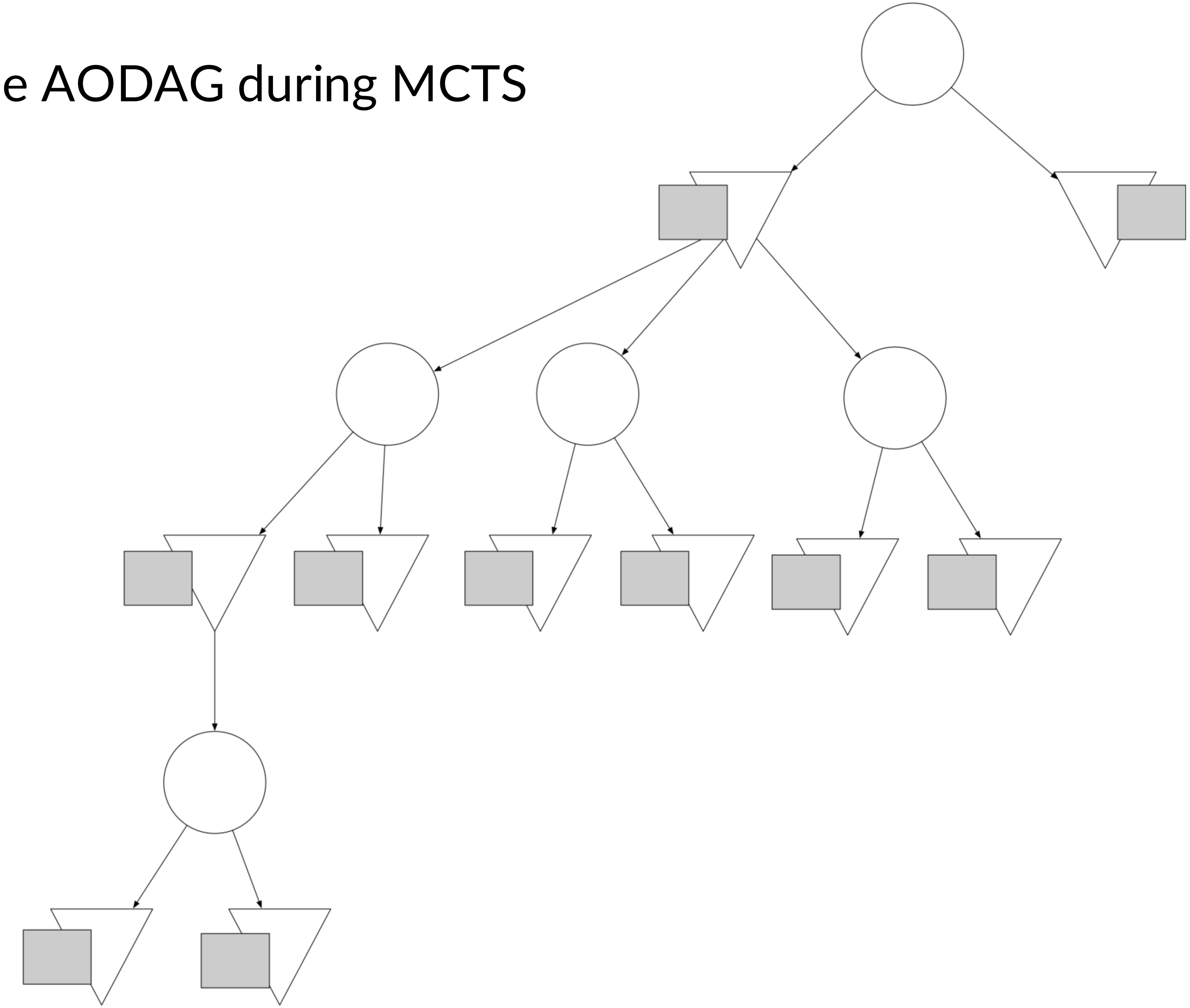
Unlike expectimax / sparse sampling, but *like* RTDP, we're going to maintain and update $\hat{Q}$ for nodes in the AODAG, rather than calculating them once and for all.

Example AODAG during MCTS



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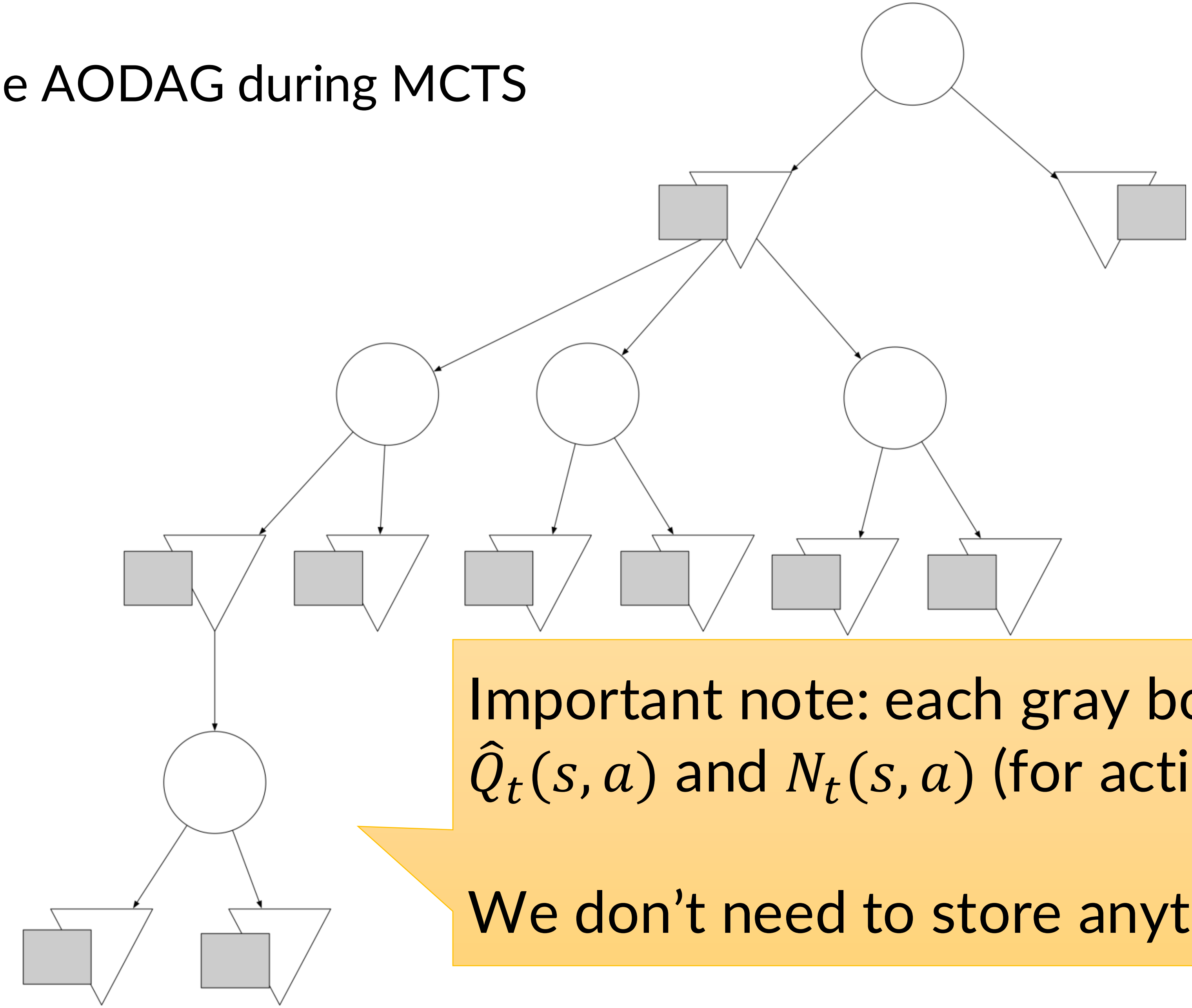

Important note: each gray box now includes *both* $\hat{Q}_t(s, a)$ and $N_t(s, a)$ (for action nodes).

We don't need to store anything at state nodes.

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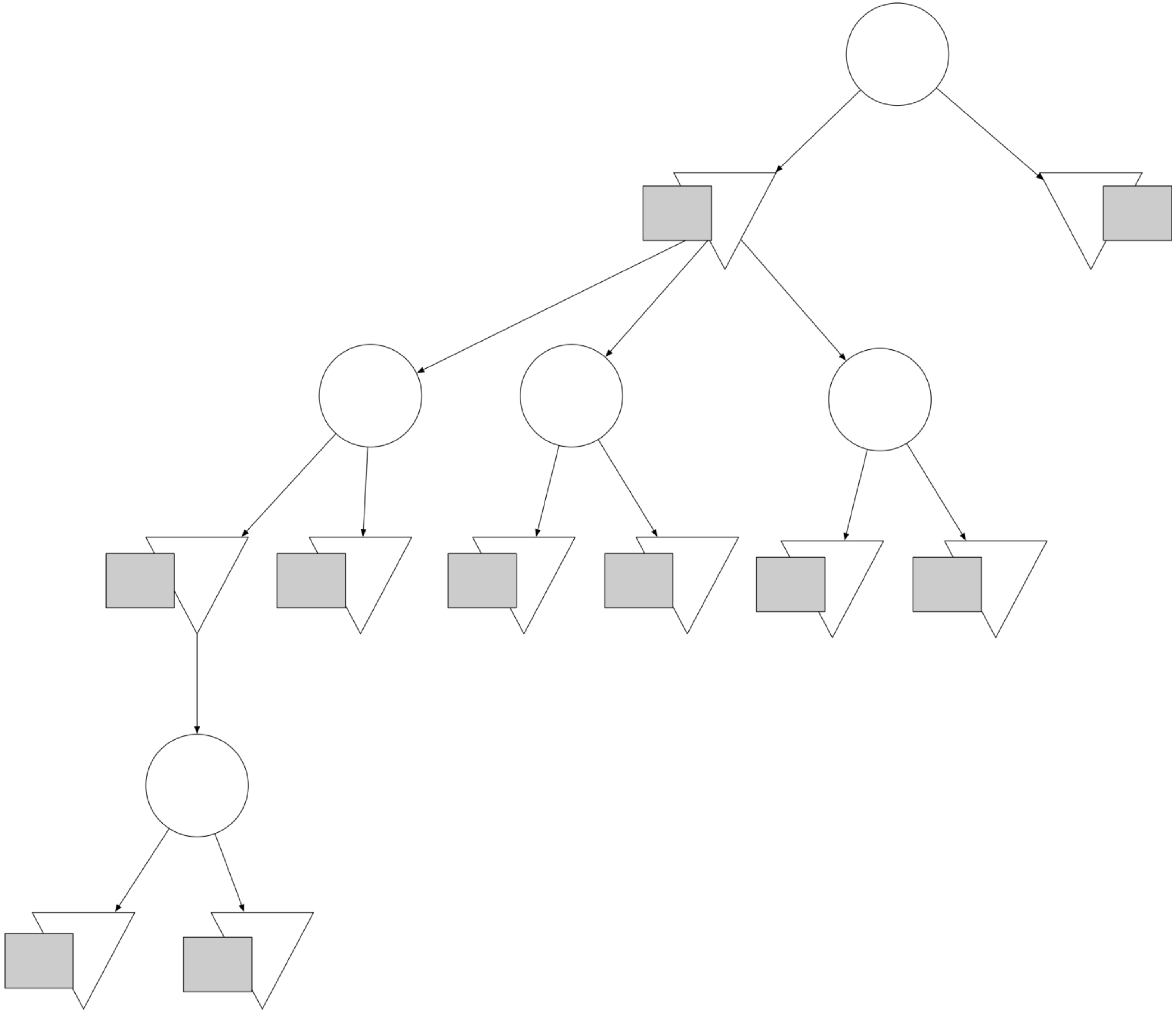
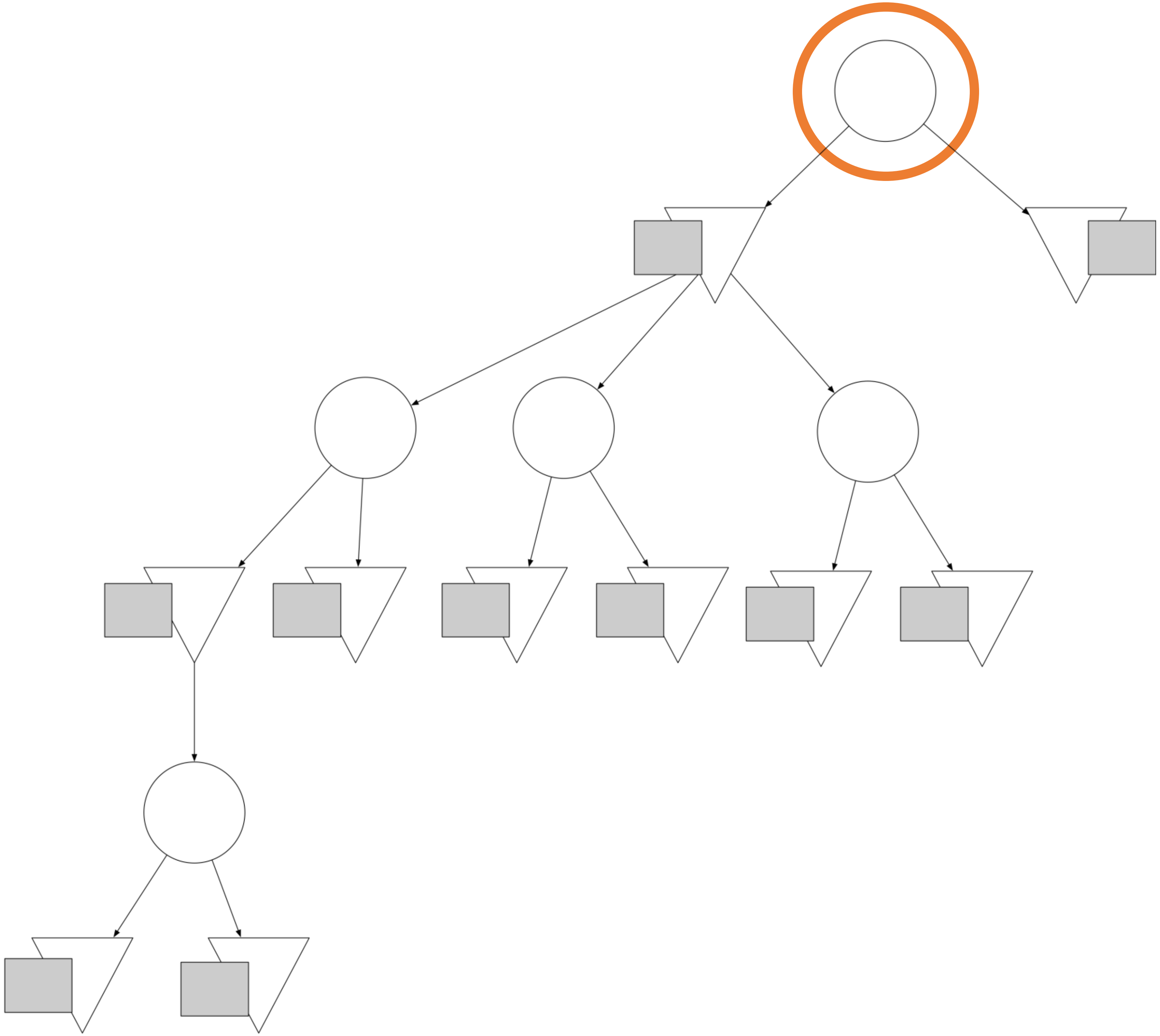
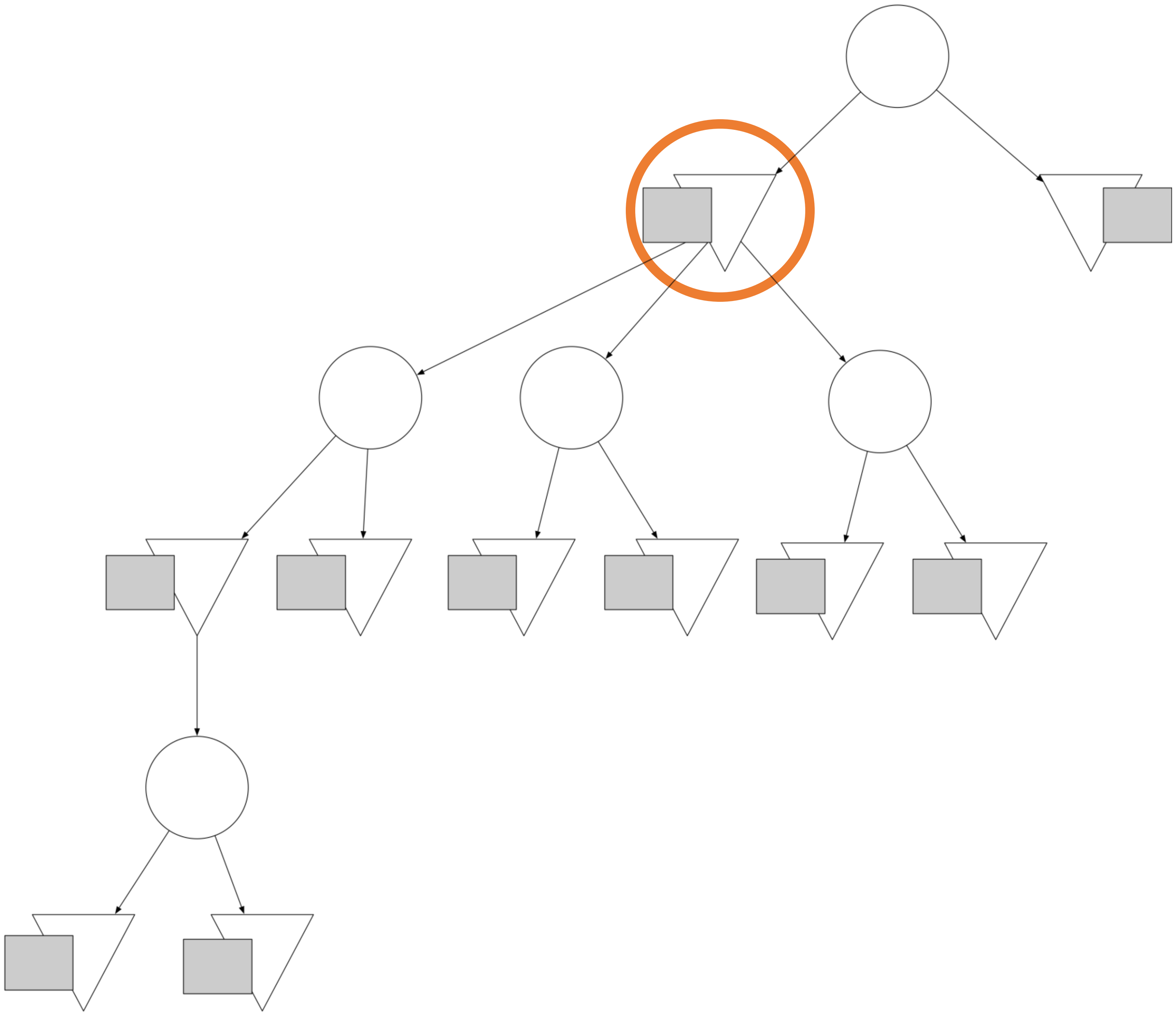
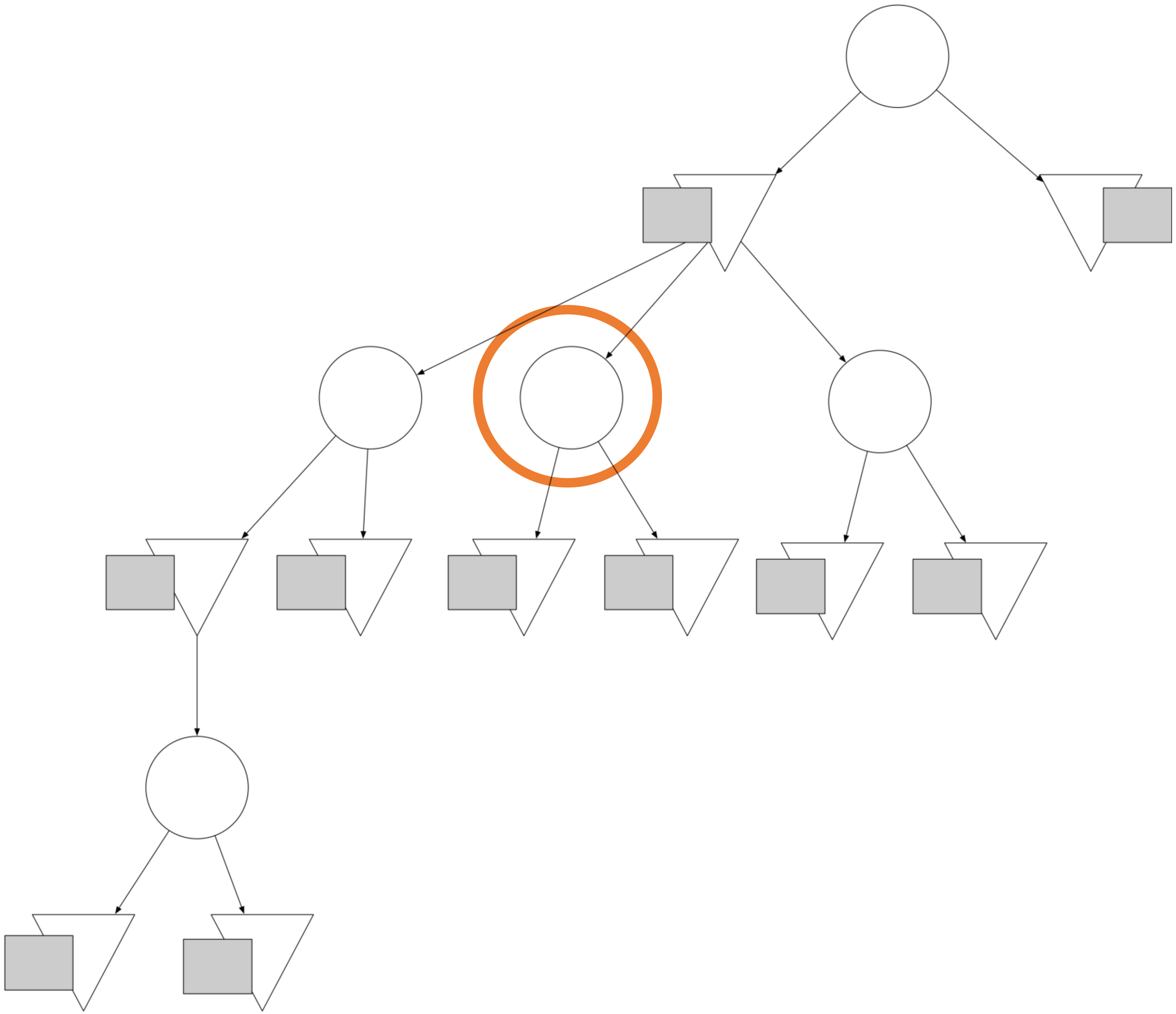
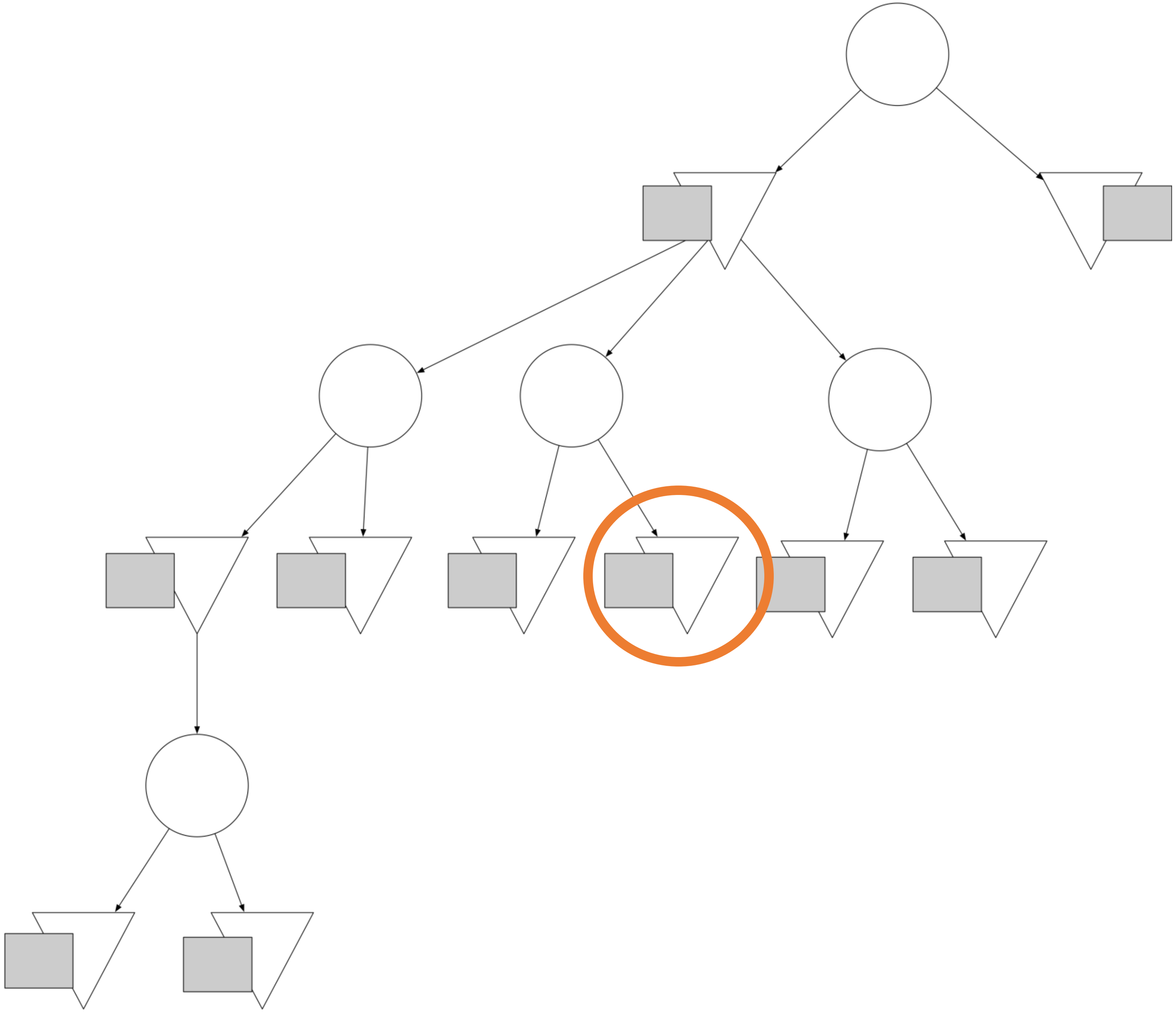
Use MAB ideas











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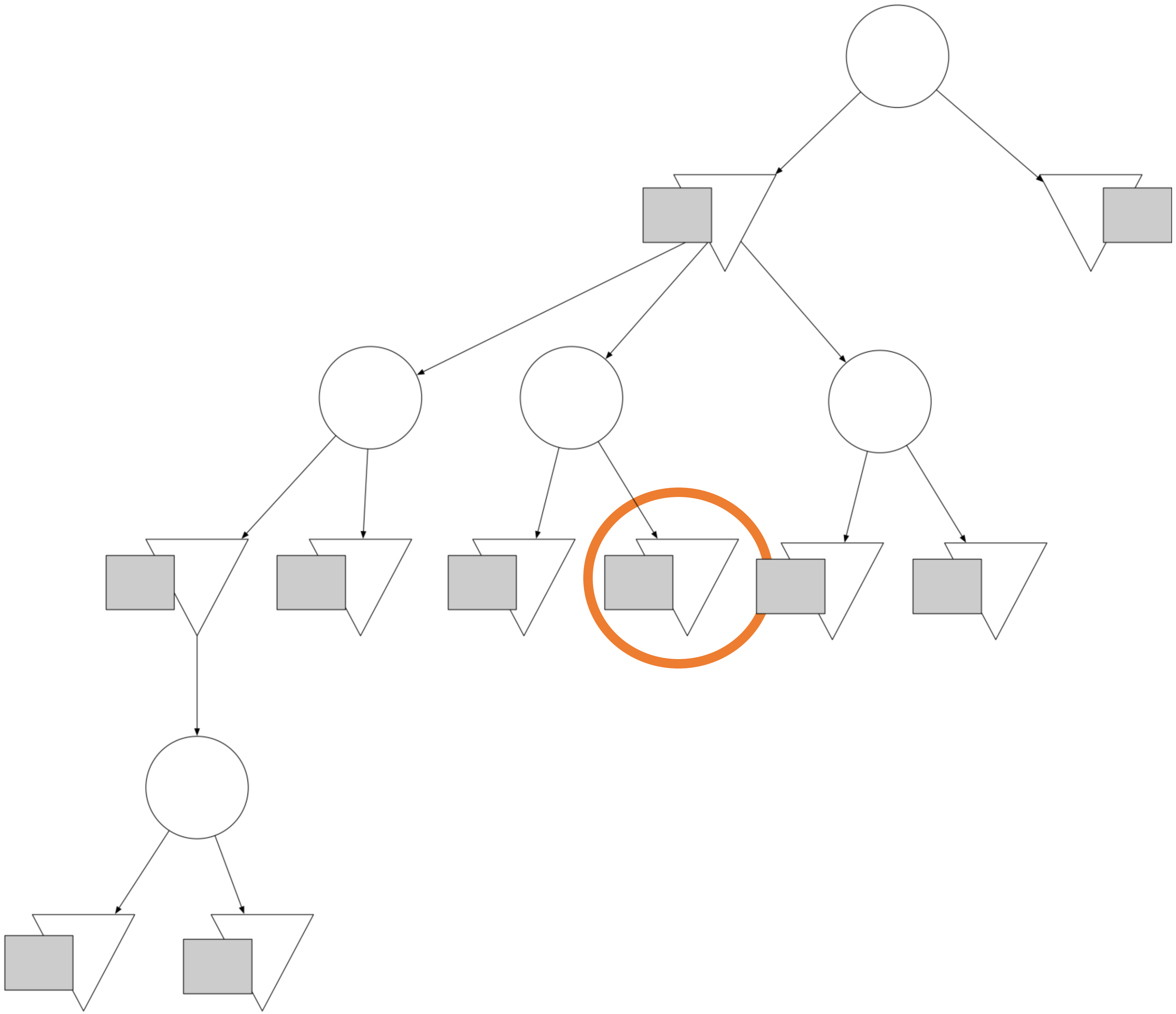

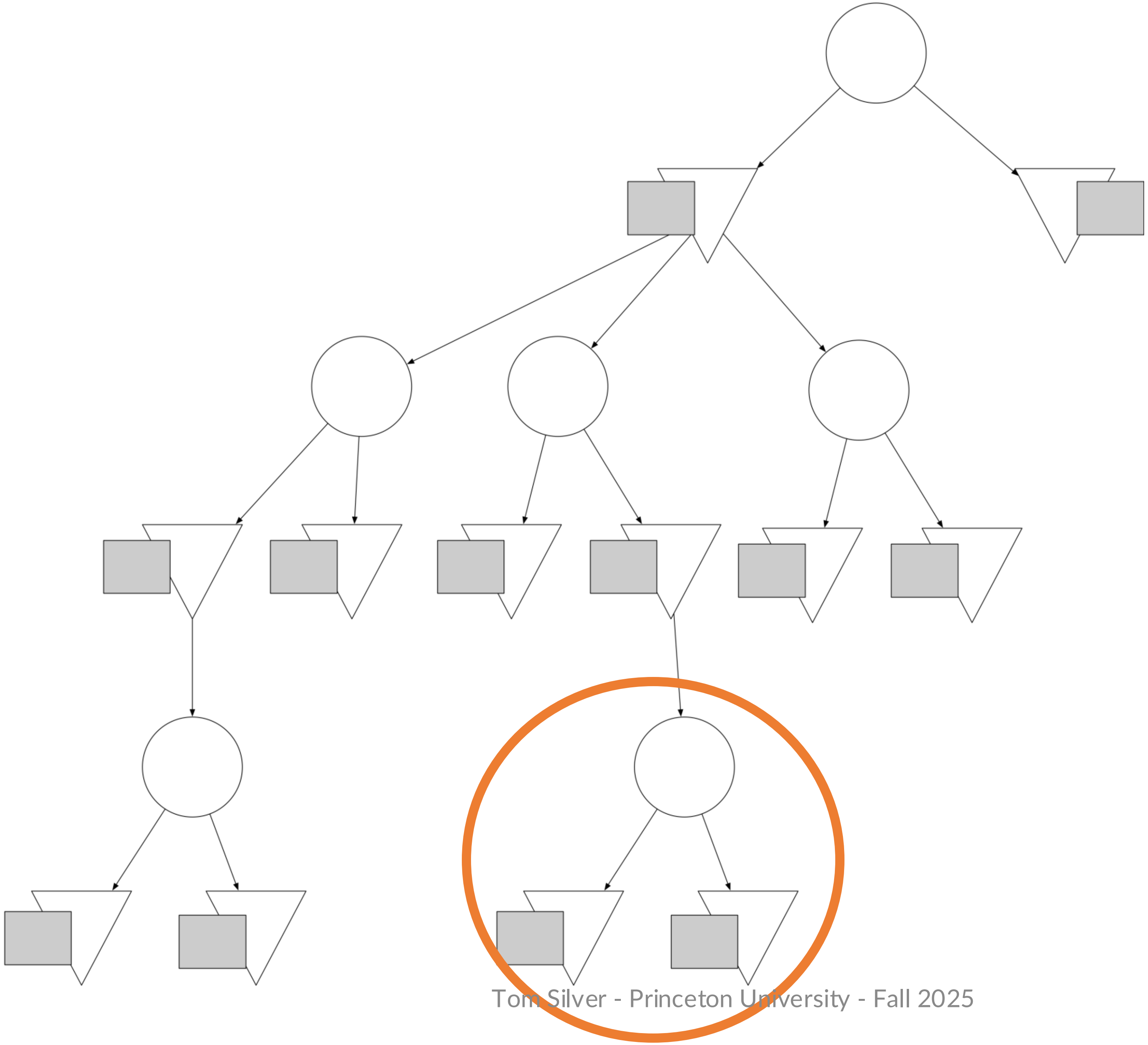

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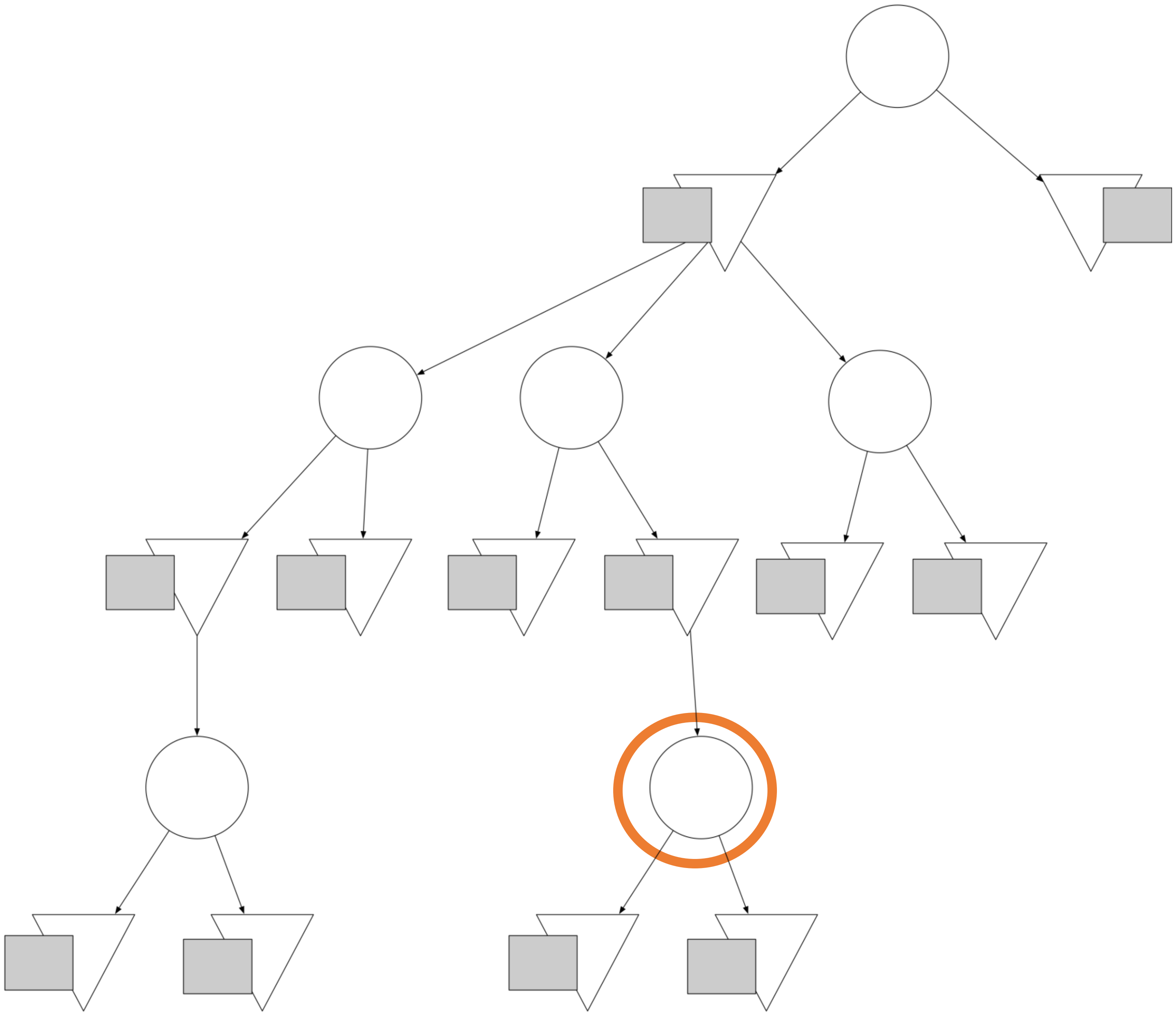

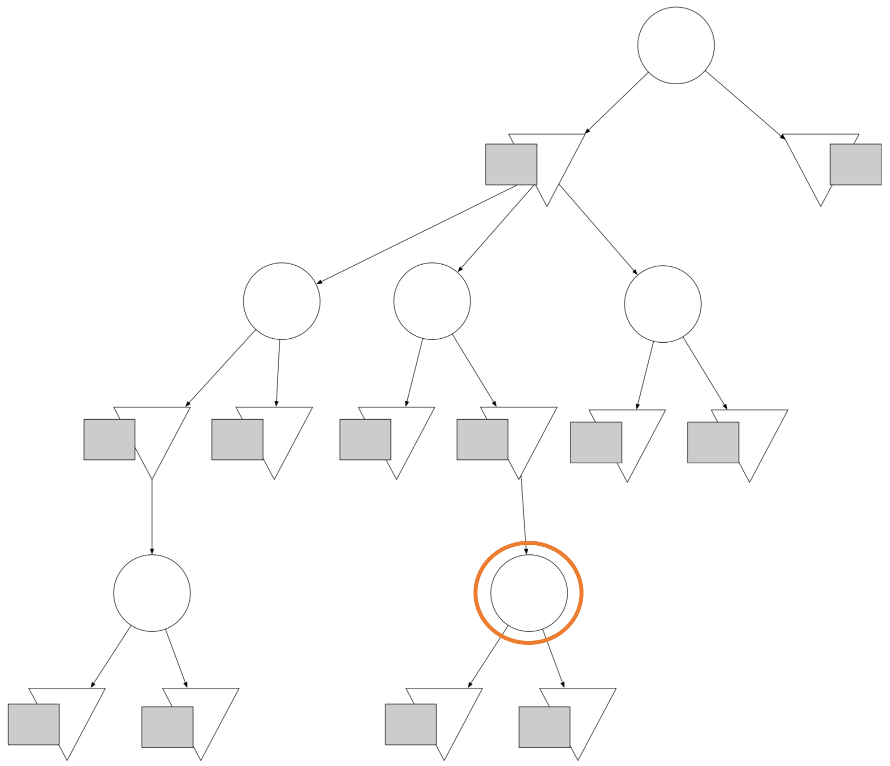

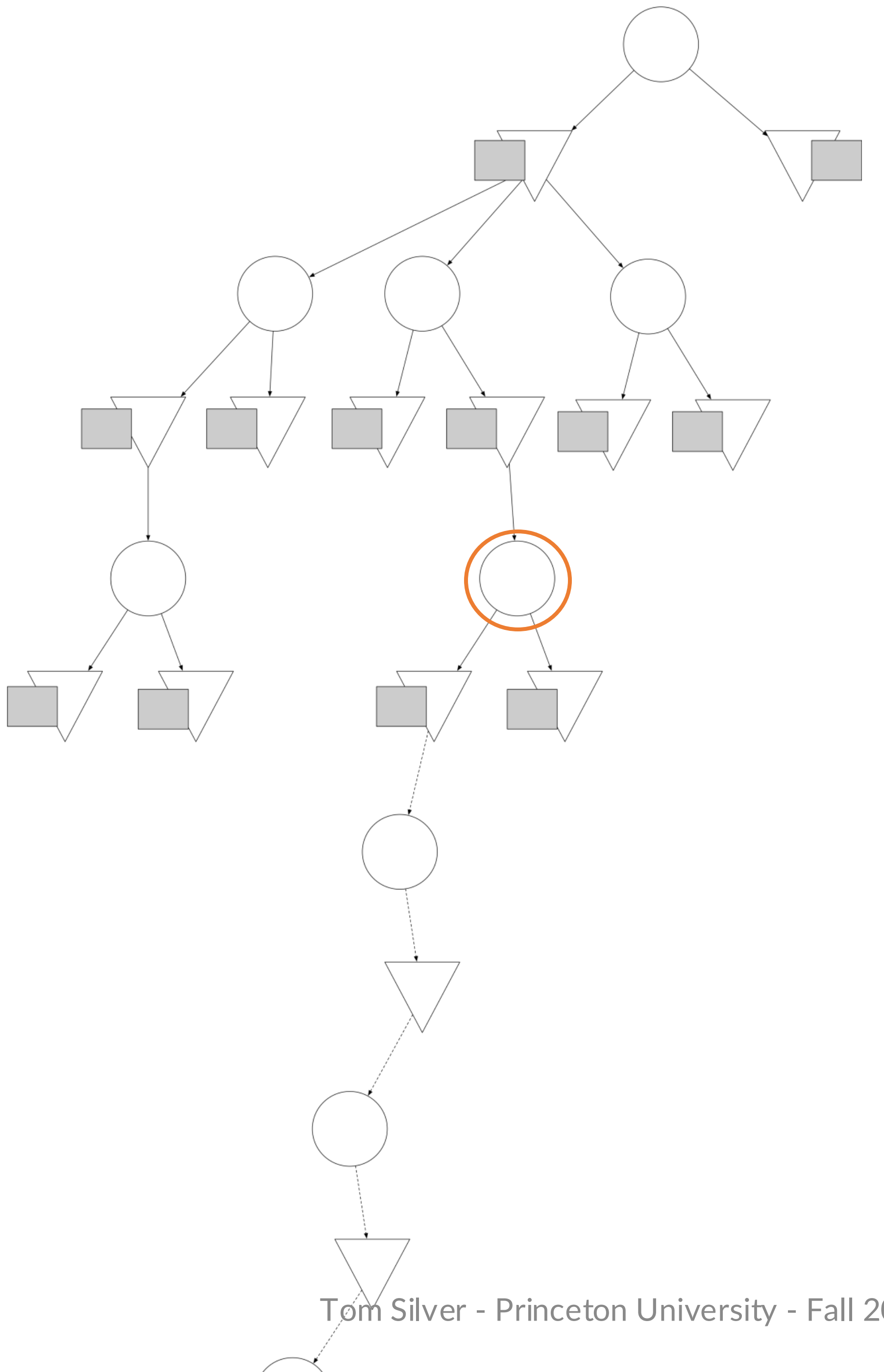

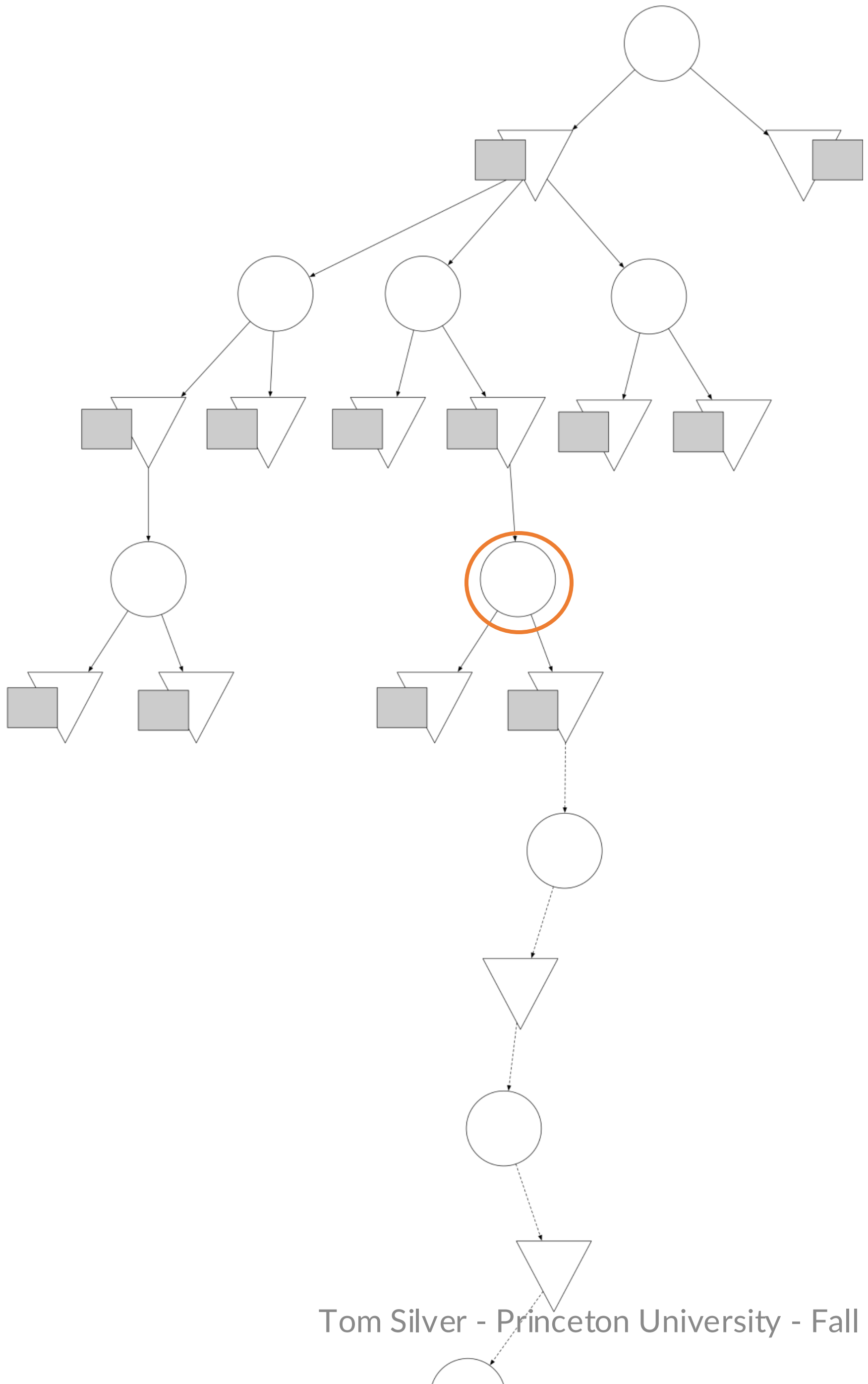

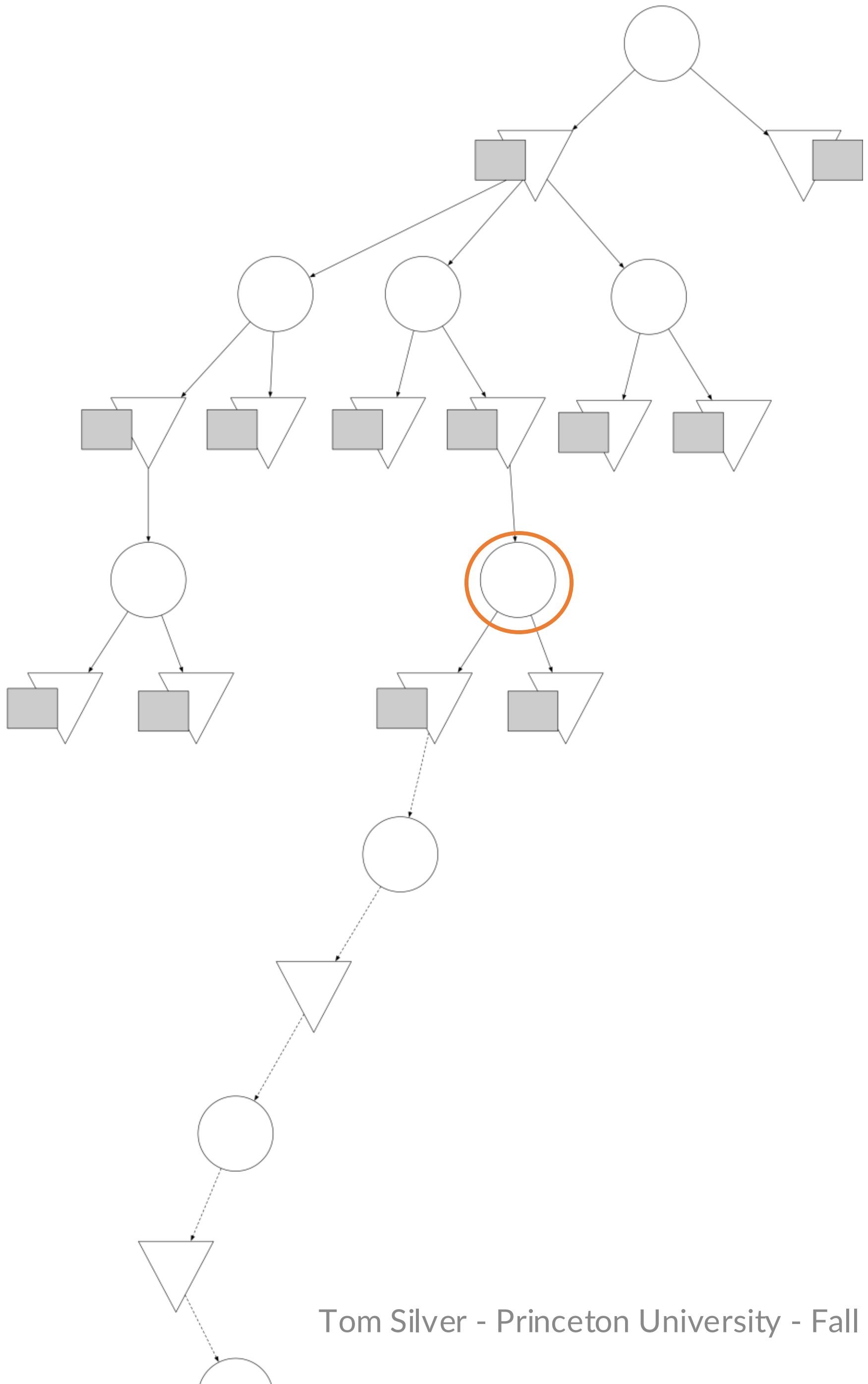

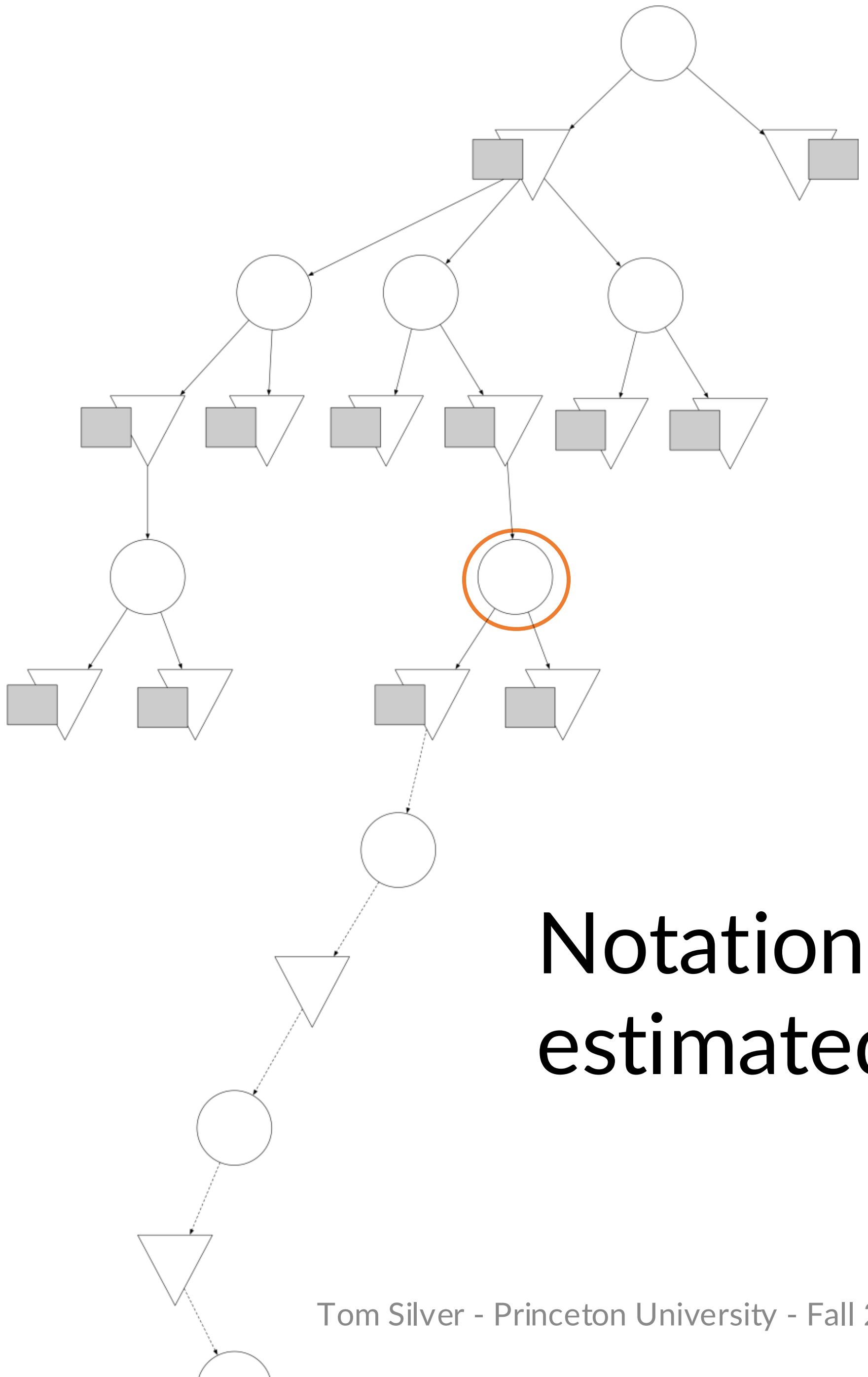

Notation: let ρ denote the estimated heuristic from rollouts.

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But if you have a heuristic, maybe use that instead!
But, good to be *admissible*.

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Not neural network backprop!

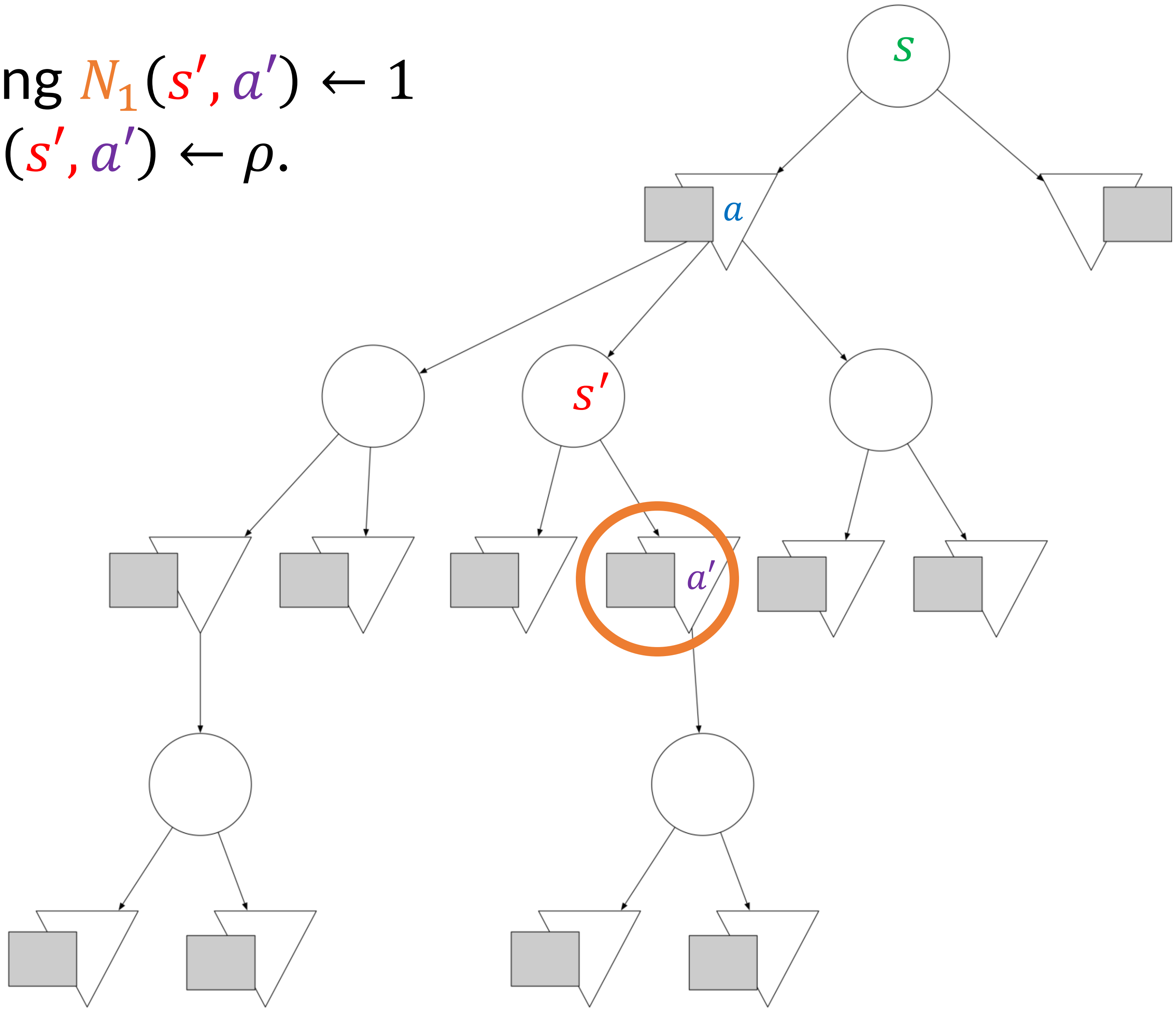
MCTS Backpropagation

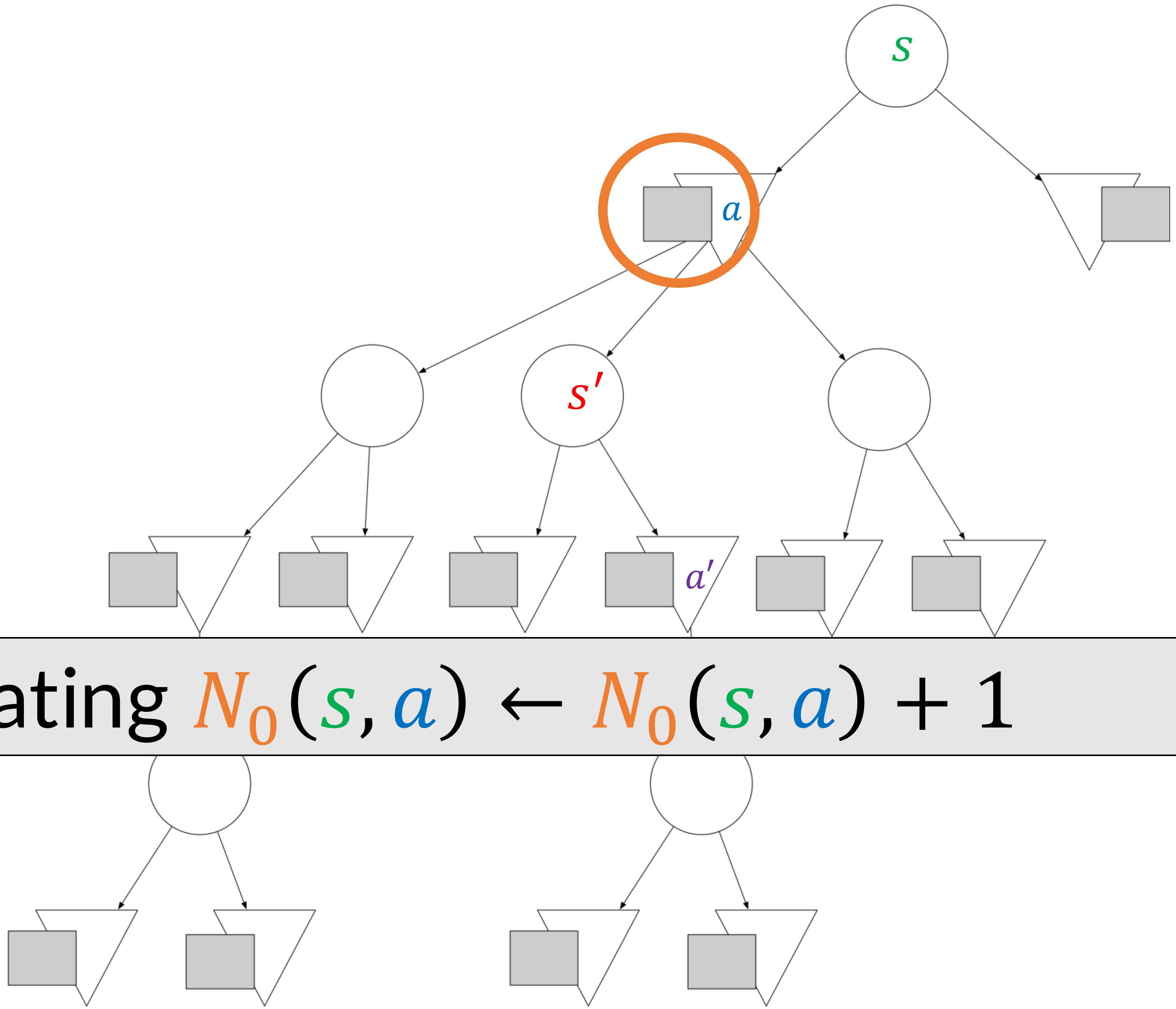
- $\widehat{Q_t}(s, a)$ will be the average of all cumulative rewards seen during planning, when starting at s at time t and taking a.

- And, $N_t(s, a)$ should be the visitation counts.

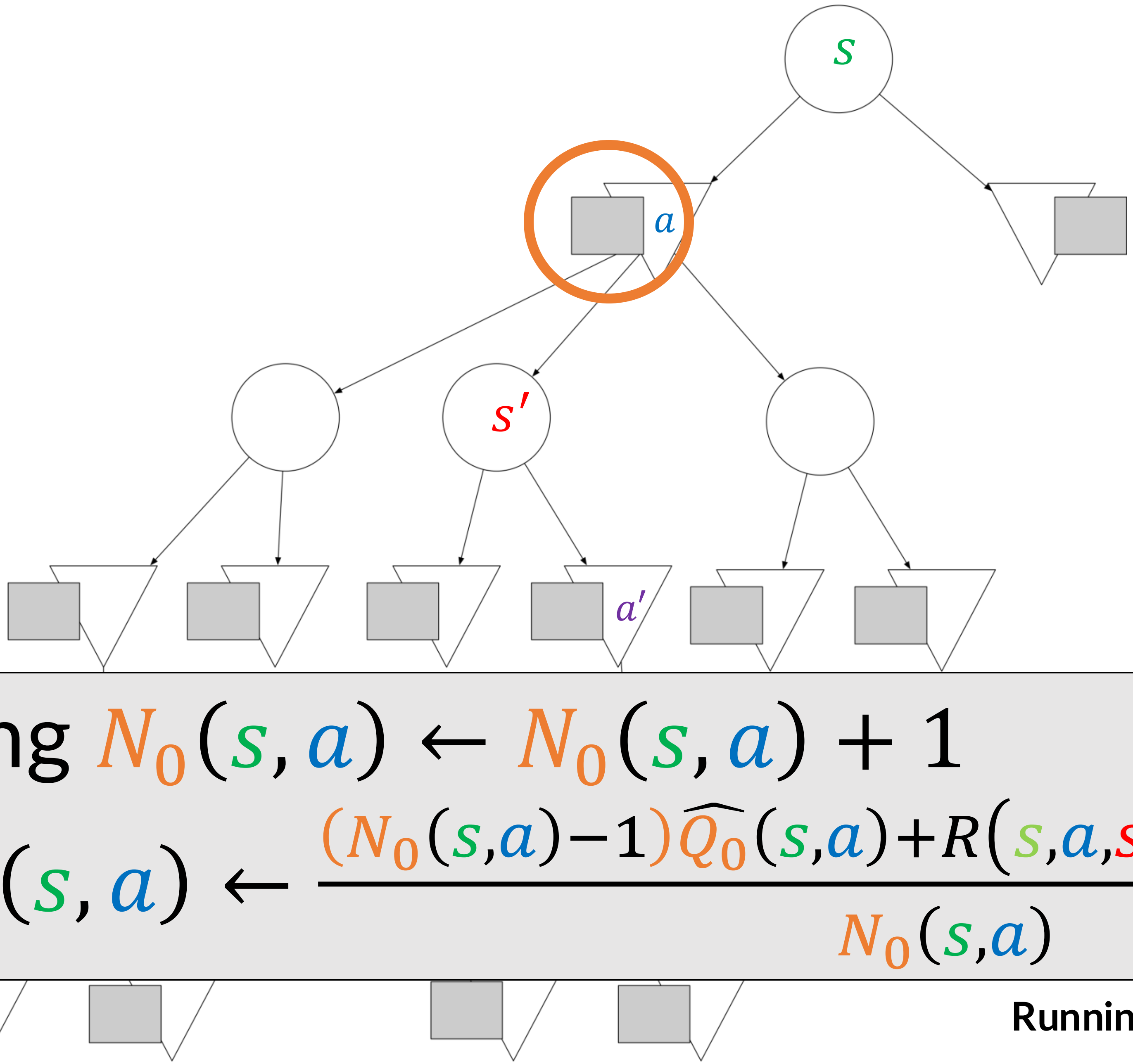
- Backpropagation: given one new trajectory, update $\hat{Q}, N$.

Updating $N_1(s', a') \leftarrow 1$ and $\widehat{Q_1}(s', a') \leftarrow \rho$.





Updating $N_0(s, a) \leftarrow N_0(s, a) + 1$



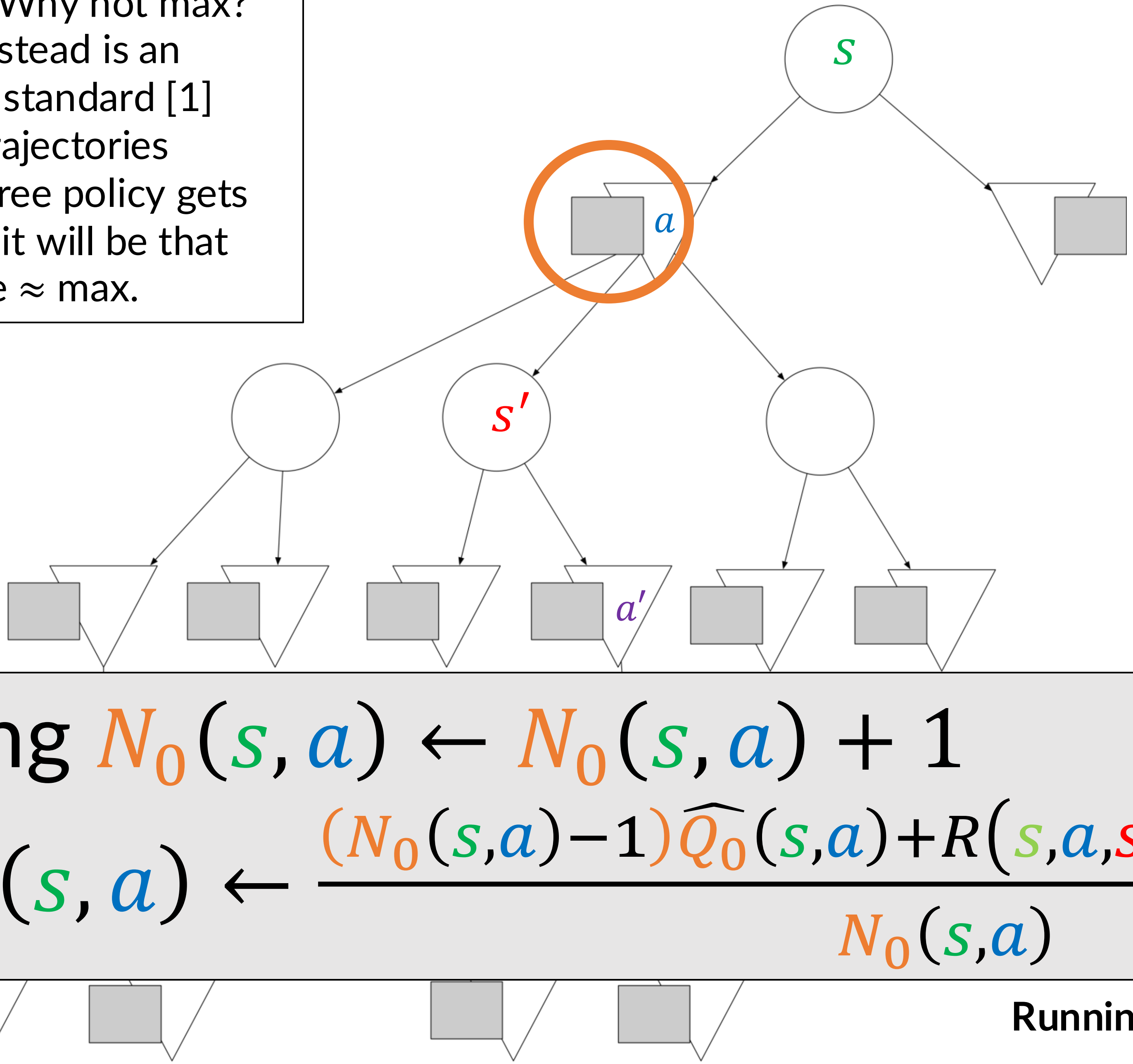
Updating $N_0(s,a) \leftarrow N_0(s,a) + 1$

and $\widehat{Q_0}(s,a) \leftarrow \dfrac{(N_0(s,a)-1)\widehat{Q_0}(s,a)+R(s,a,s')+\gamma\widehat{Q_1}(s',a')}{N_0(s,a)}$

Running average!

Running average? Why not max?

- Taking a max instead is an option, but less standard [1]
- As number of trajectories increases, and tree policy gets more exploit-y, it will be that running average ≈ max.



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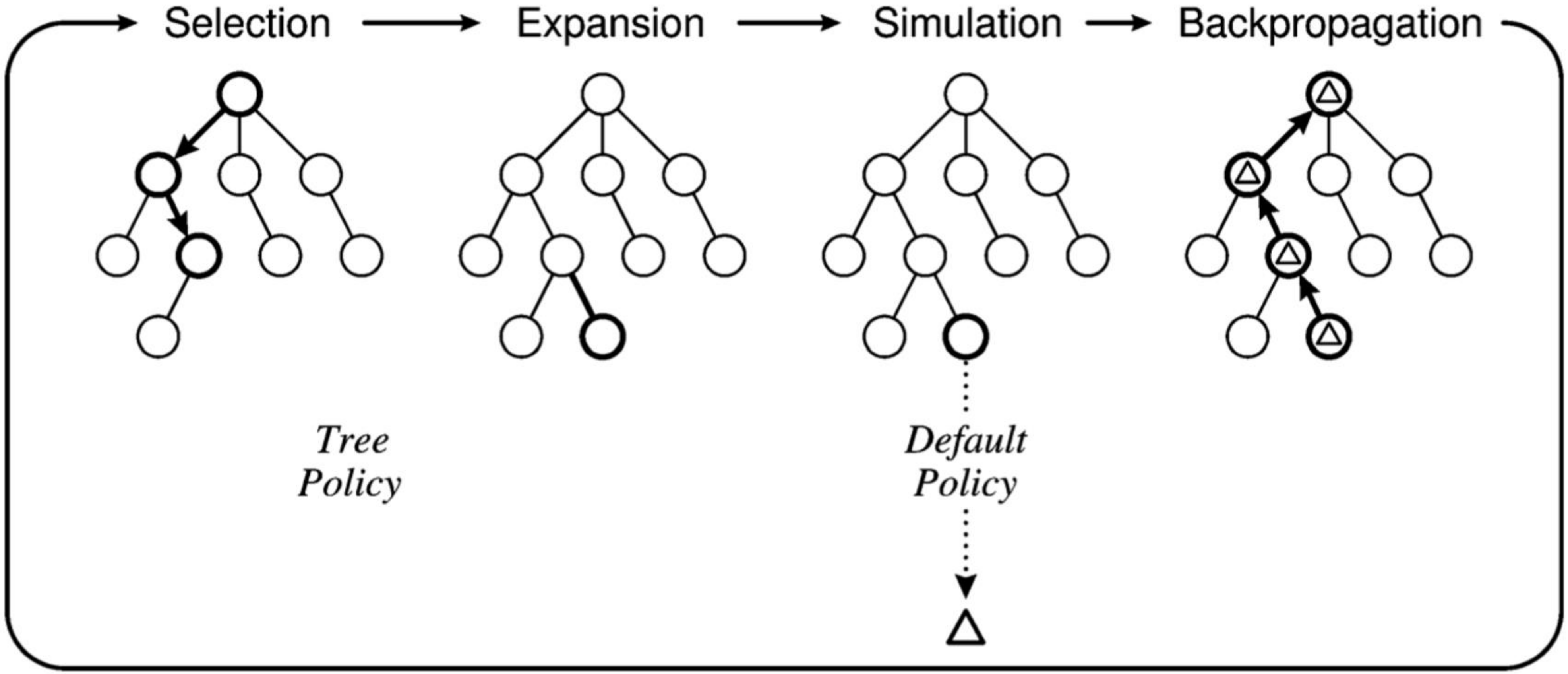
Running average!

[1] https://www.cs.utexas.edu/~pstone/Papers/bib2html-links/ICML16-khandelwal.slides.pdf

MCTS Summary



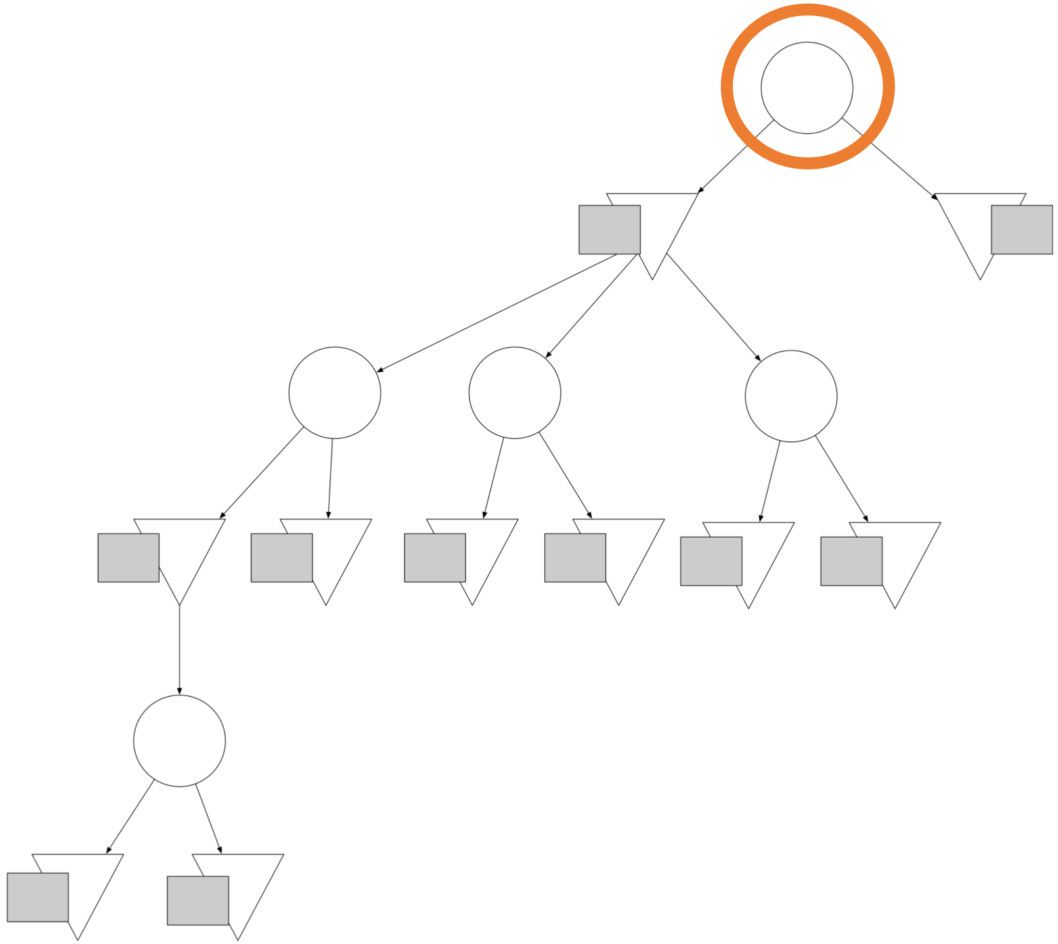
“A Survey of Monte Carlo Tree Search Methods.” Browne et al. (2012).

MCTS$(s_0, \mathcal{S}, \mathcal{A}, P, R, \gamma)$

`Q` $=$ `dict()` // Estimate for $Q_t(s, a)$
`N` $=$ `dict()` // Visitation counts $N_t(s, a)$
repeat until time runs out
    Simulate$(s_0, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, 0)$ // Updates `Q` and `N`
return $\text{argmax}_a\, \texttt{Q}(0, s_0, a)$

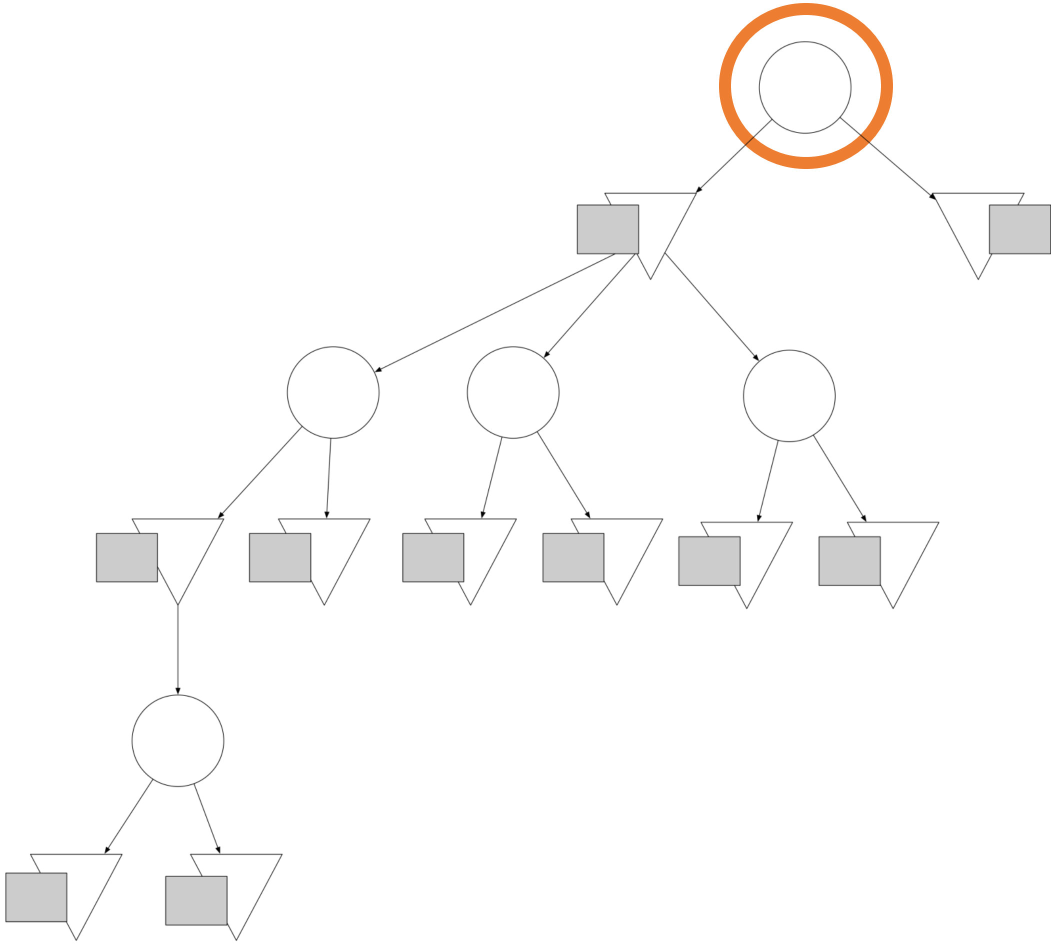
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
    for $\mathtt{a} \in \mathcal{A}$
        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



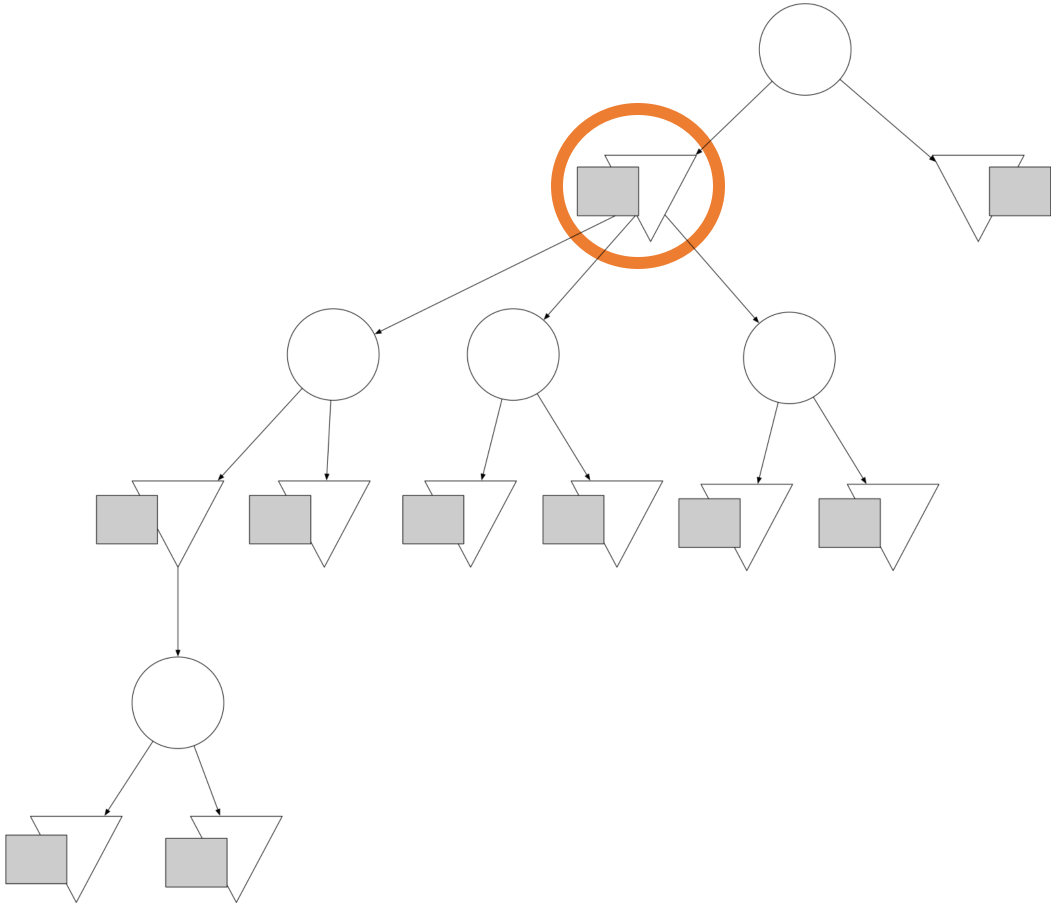
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



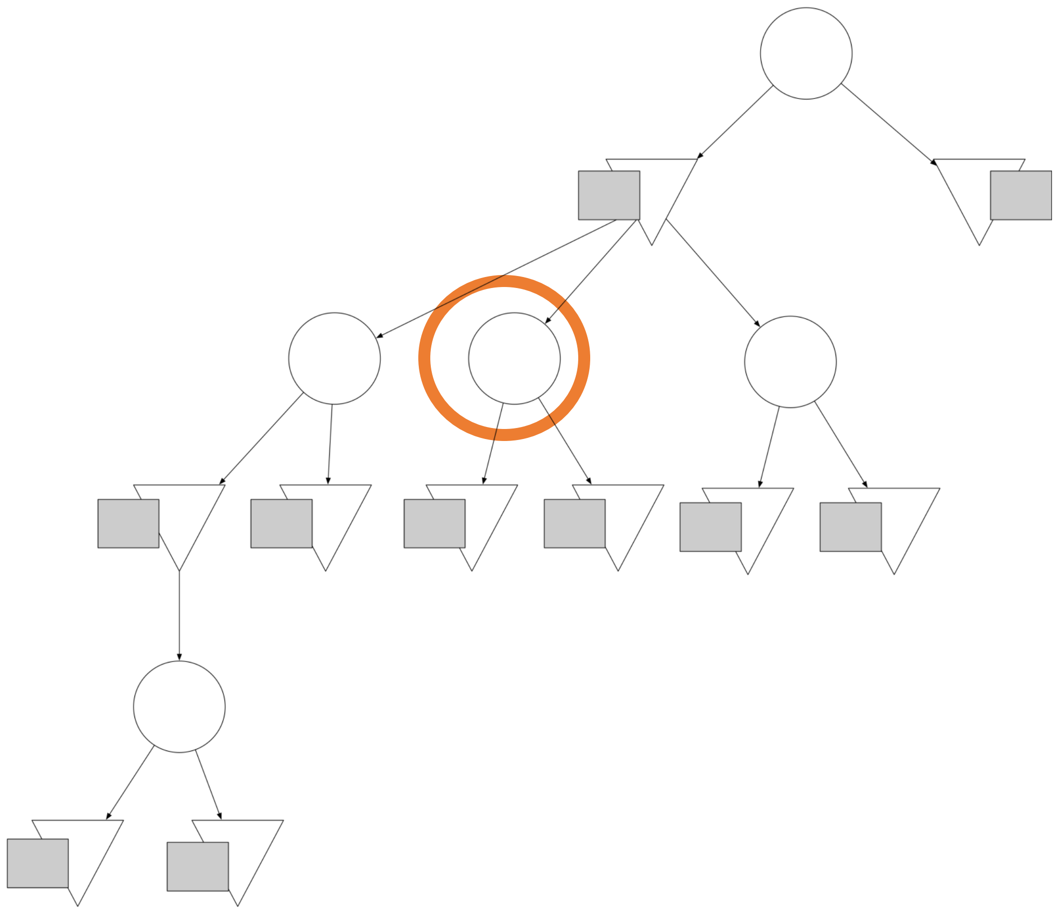
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$ **False**
    for $\mathtt{a} \in \mathcal{A}$
        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



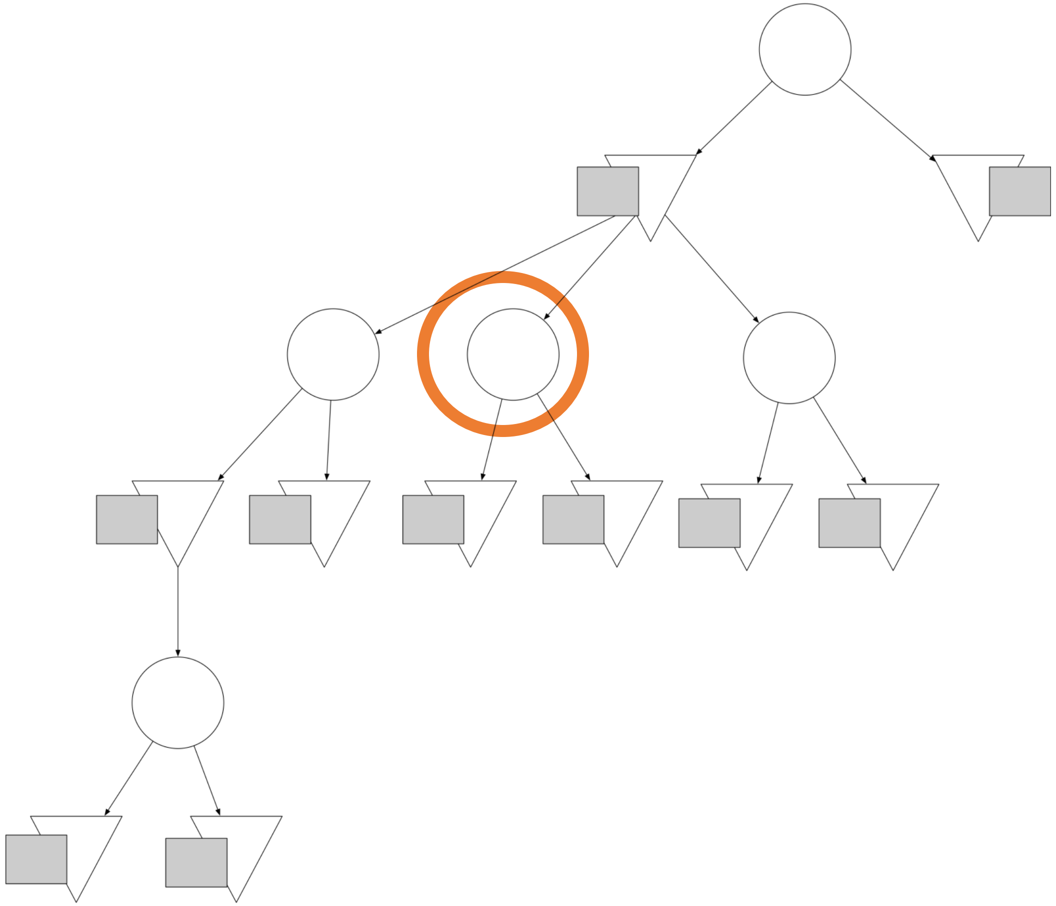
$\textsc{Simulate}(\texttt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, \texttt{t})$

// Base case: we've never visited this state at this depth before
if $(\texttt{t}, \texttt{s}, \texttt{a}) \notin \texttt{N}$ for an arbitrary $\texttt{a} \in \mathcal{A}$
    for $\texttt{a} \in \mathcal{A}$
        $\texttt{N}(\texttt{t}, \texttt{s}, \texttt{a}) = 0$
        $\texttt{Q}(\texttt{t}, \texttt{s}, \texttt{a}) = 0$
    return $\textsc{EstimateHeuristic}(\texttt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\boxed{\texttt{a} = \textsc{Explore}(\texttt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, \texttt{t})}$ // differs between MCTS algs
$\texttt{ns} \sim \mathrm{P}(\cdot \mid \texttt{s}, \texttt{a})$
$\texttt{qtsa} = R(\texttt{s}, \texttt{a}, \texttt{ns}) + \gamma \textsc{Simulate}(\texttt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, \texttt{t} + 1)$
$\texttt{N}(\texttt{t}, \texttt{s}, \texttt{a}) = \texttt{N}(\texttt{t}, \texttt{s}, \texttt{a}) + 1$
$\texttt{Q}(\texttt{t}, \texttt{s}, \texttt{a}) = \frac{(\texttt{N}(\texttt{t},\texttt{s},\texttt{a}) - 1)\texttt{Q}(\texttt{t},\texttt{s},\texttt{a}) + \texttt{qtsa}}{\texttt{N}(\texttt{t},\texttt{s},\texttt{a})}$
return $\texttt{Q}(\texttt{t}, \texttt{s}, \texttt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\boxed{\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})}$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before

if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$

 for $\mathtt{a} \in \mathcal{A}$

  $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

  $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

 return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts

$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs

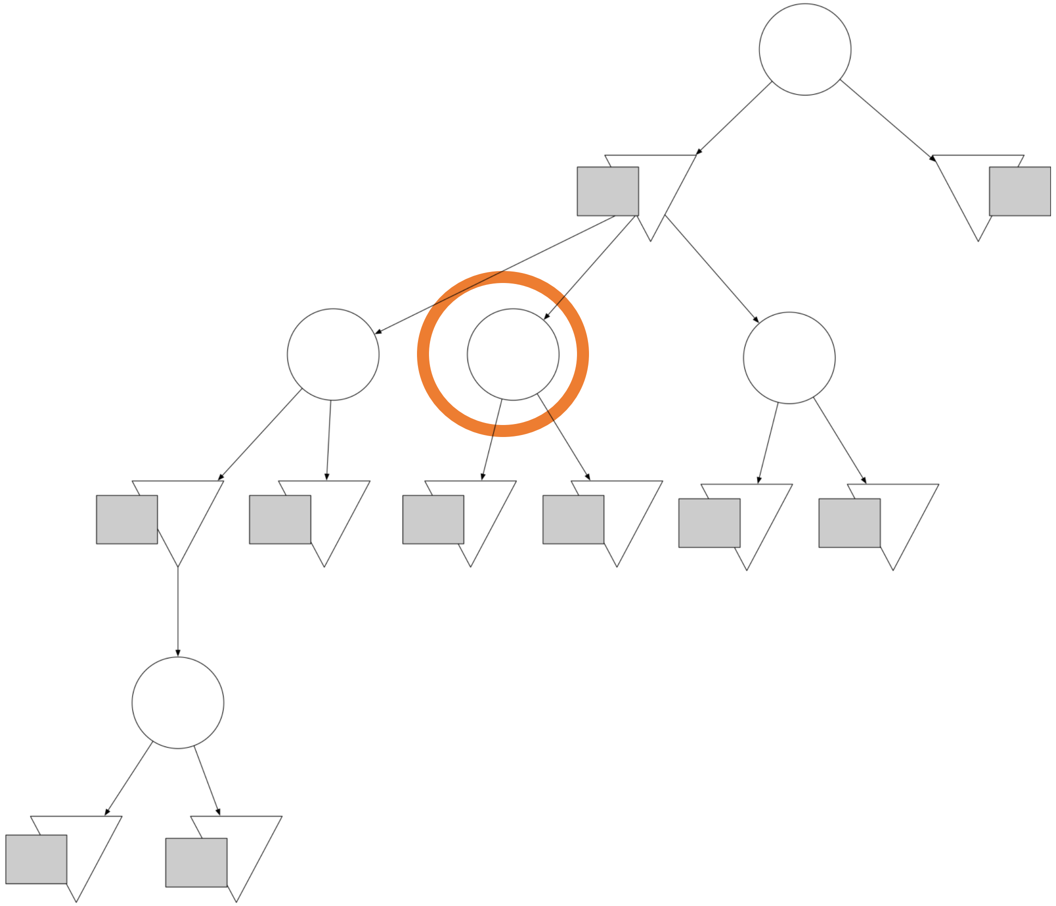
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$

$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \boxed{\textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t}+1)}$

$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$

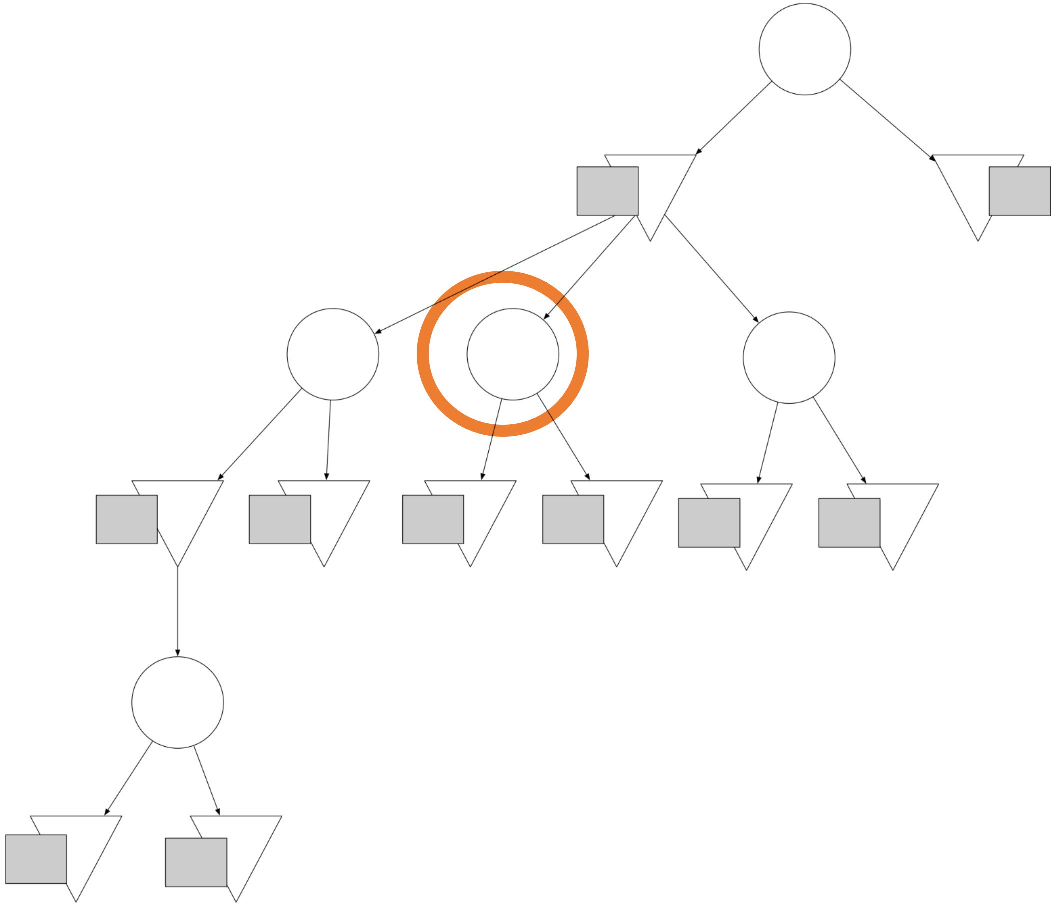
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})-1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a})+\mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$

return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



$\boxed{\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})}$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before

$\boxed{\textbf{if } (\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N} \text{ for an arbitrary } \mathtt{a} \in \mathcal{A}}$ **False**

    for $\mathtt{a} \in \mathcal{A}$

        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

    return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts

$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs

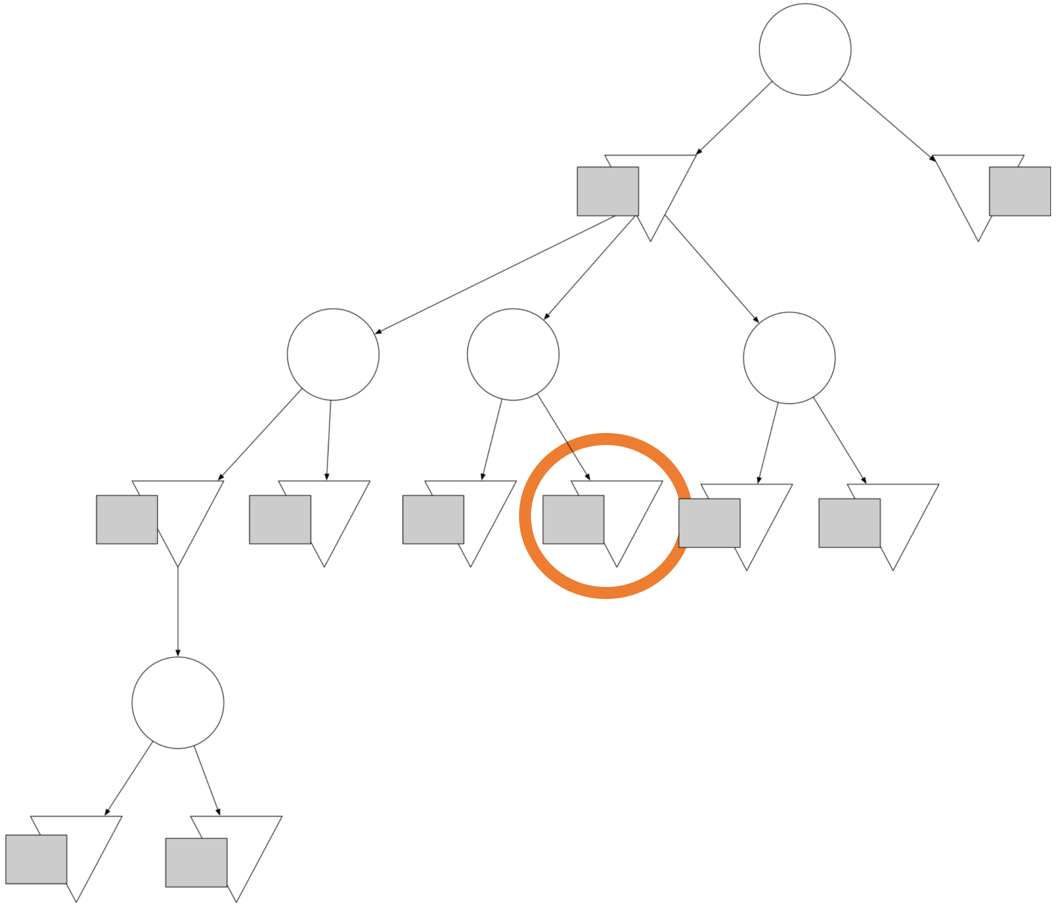
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$

$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$

$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$

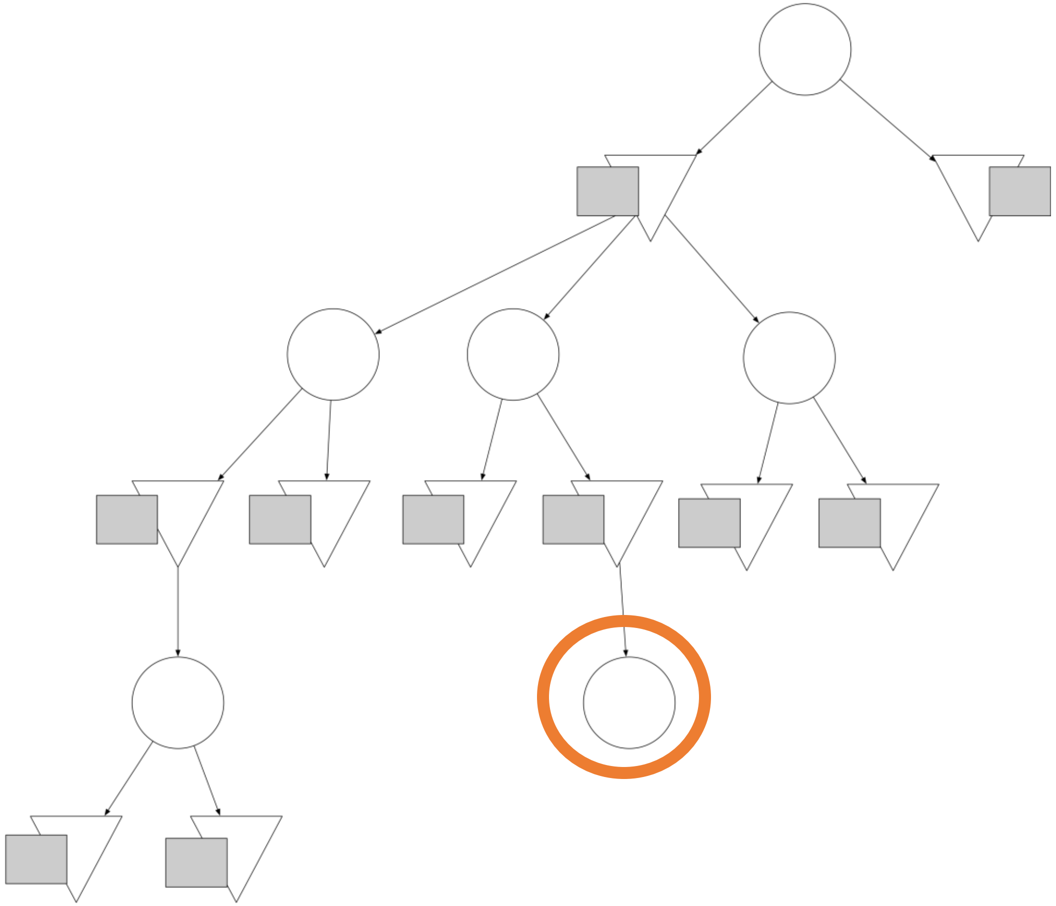
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$

return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



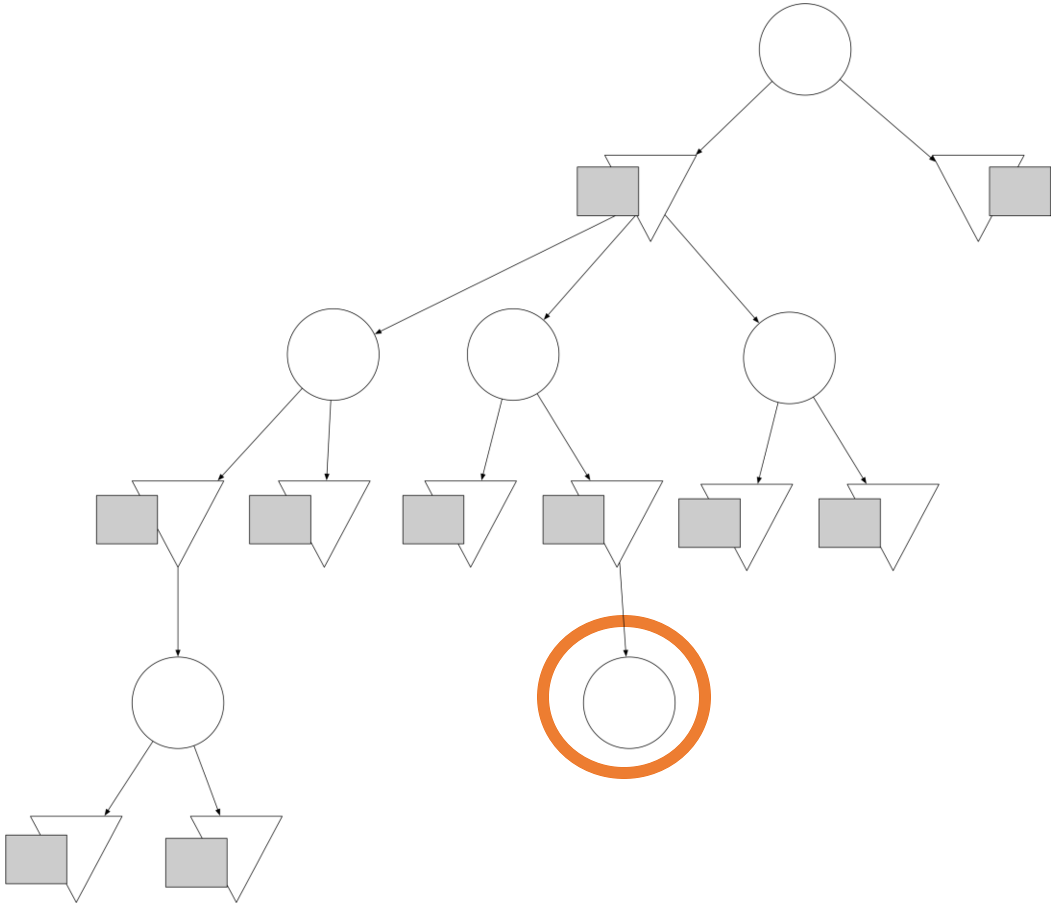
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\boxed{\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})}$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



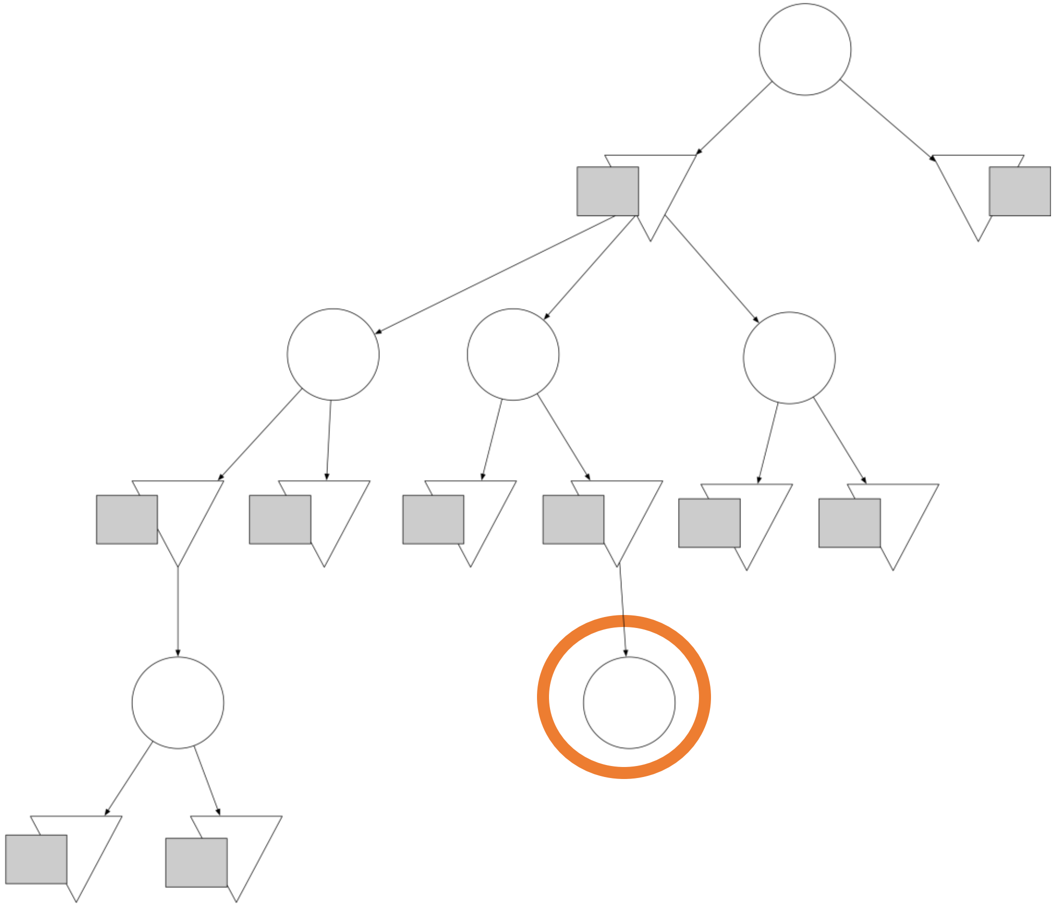
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
    for $\mathtt{a} \in \mathcal{A}$
        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\boxed{\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})}$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



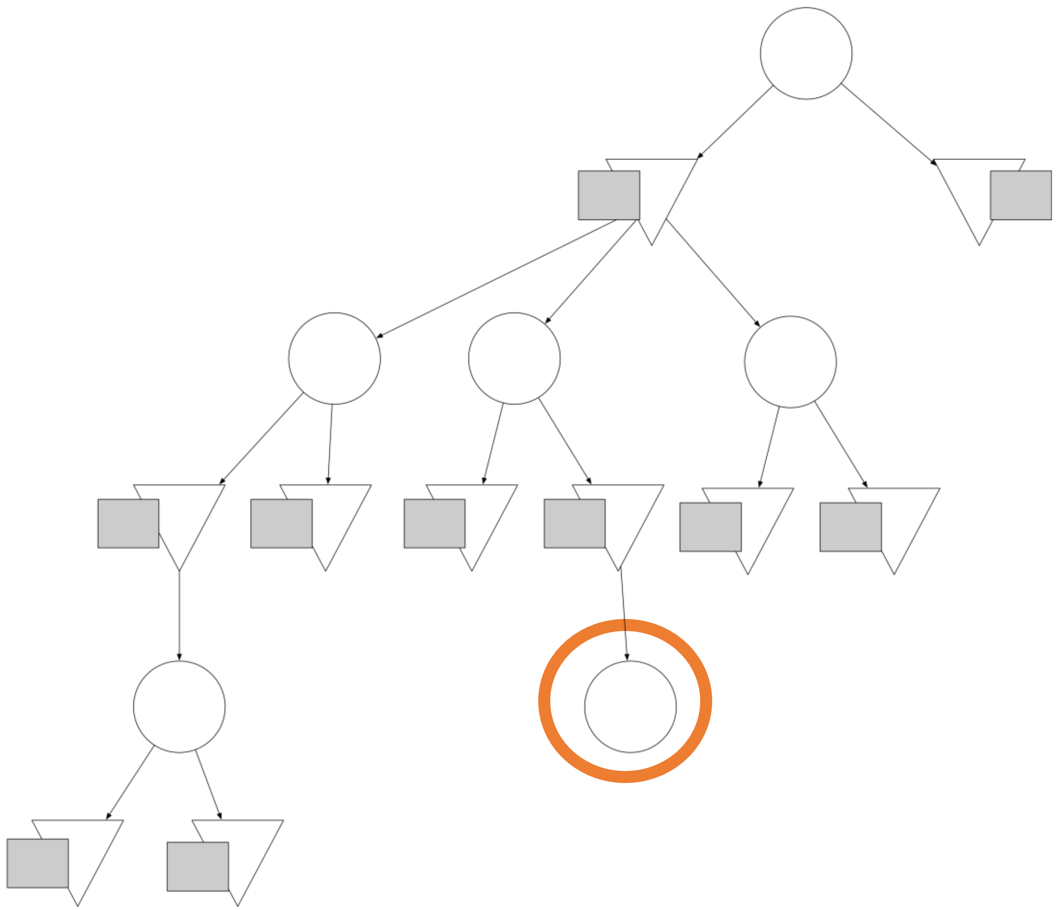
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \boxed{\textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t}+1)}$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})-1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a})+\mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$

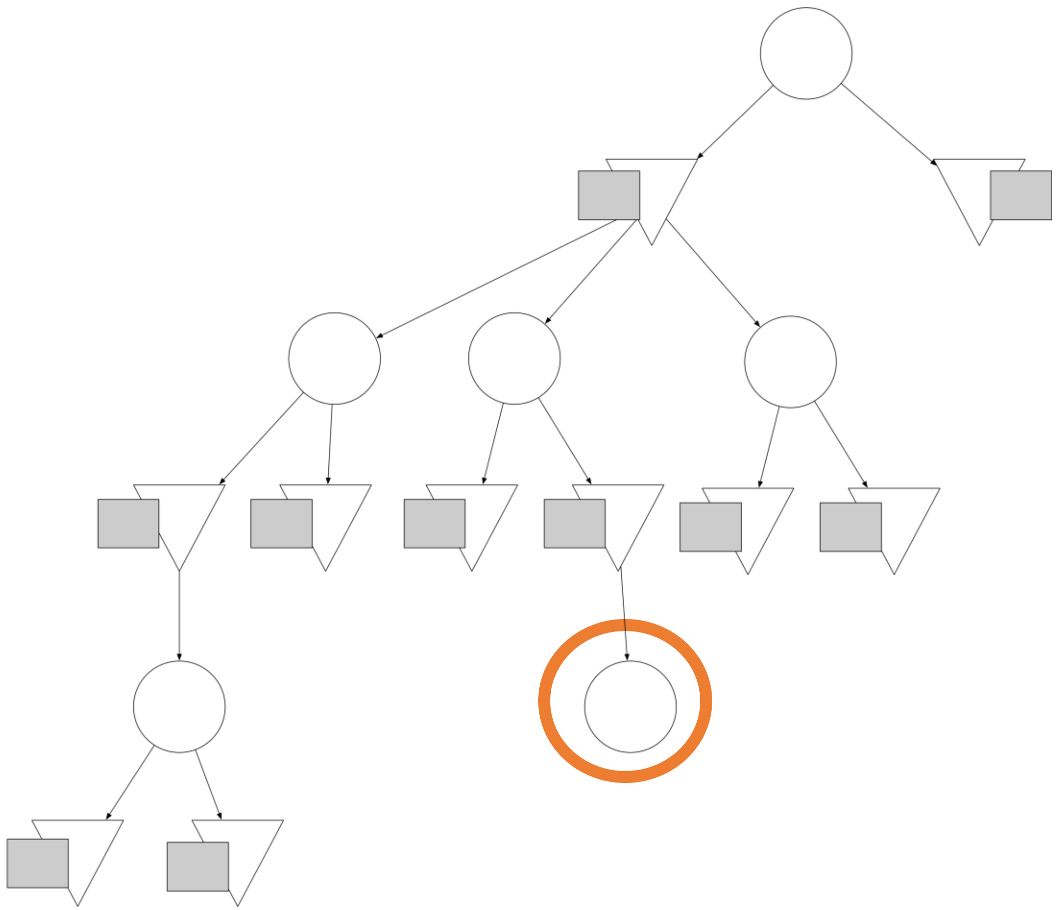


$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \boxed{\textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t}+1)}$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})-1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a})+\mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$

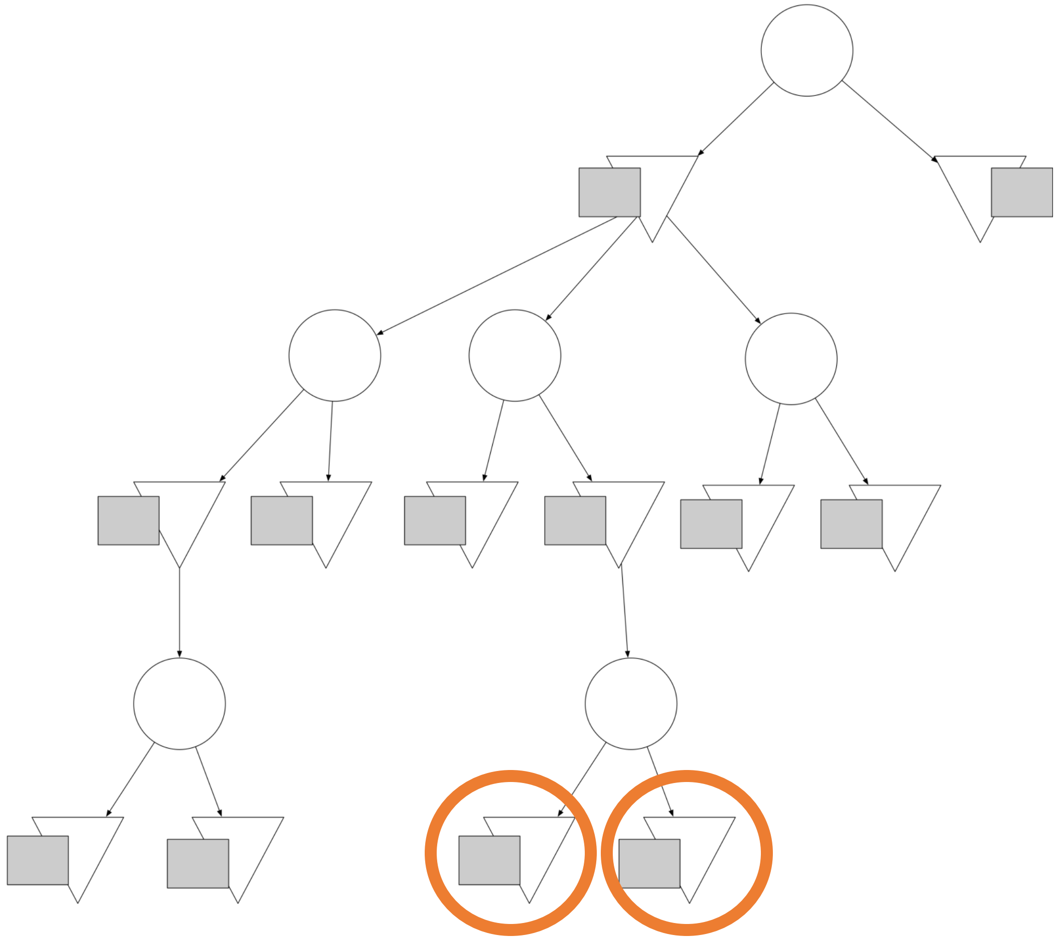


```
SIMULATE(s, 𝒮, 𝒜, P, R, γ, Q, N, t)
// Base case: we've never visited this state at this depth before
if (t, s, a) ∉ N for an arbitrary a ∈ 𝒜
    for a ∈ 𝒜
        N(t, s, a) = 0
        Q(t, s, a) = 0
    return ESTIMATEHEURISTIC(s, 𝒮, 𝒜, P, R, γ) // Run random rollouts
a = EXPLORE(s, 𝒮, 𝒜, P, R, γ, Q, N, t) // differs between MCTS algs
ns ~ P(· | s, a)
qtsa = R(s, a, ns) + γSIMULATE(ns, 𝒮, 𝒜, P, R, γ, Q, N, t + 1)
N(t, s, a) = N(t, s, a) + 1
Q(t, s, a) = ((N(t,s,a) − 1)Q(t,s,a) + qtsa) / N(t,s,a)
return Q(t, s, a)
```



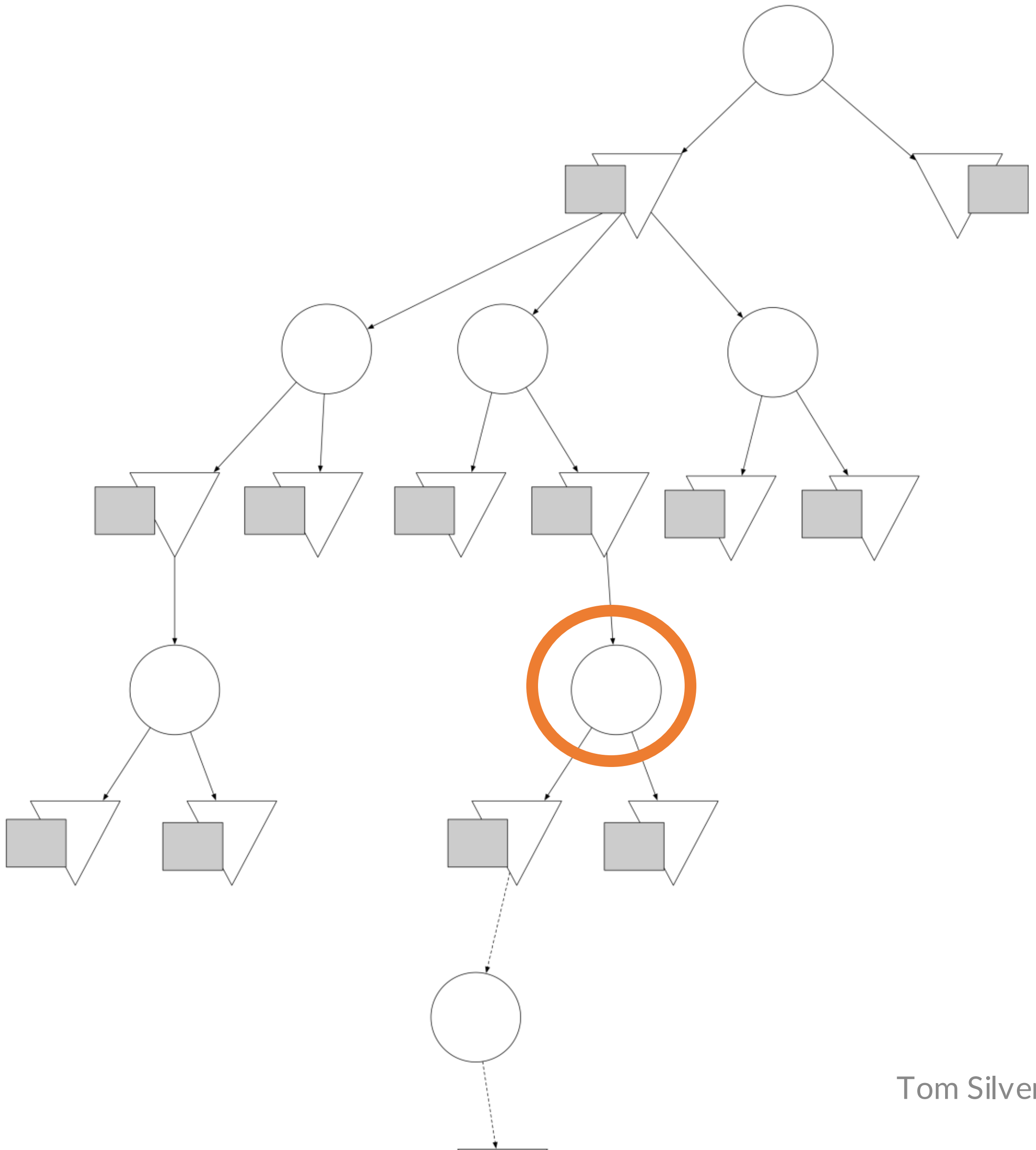
SIMULATE$(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$ **True**
    for $\mathtt{a} \in \mathcal{A}$
        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    return ESTIMATEHEURISTIC$(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} =$ EXPLORE$(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma$SIMULATE$(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t}+1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})-1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a})+\mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



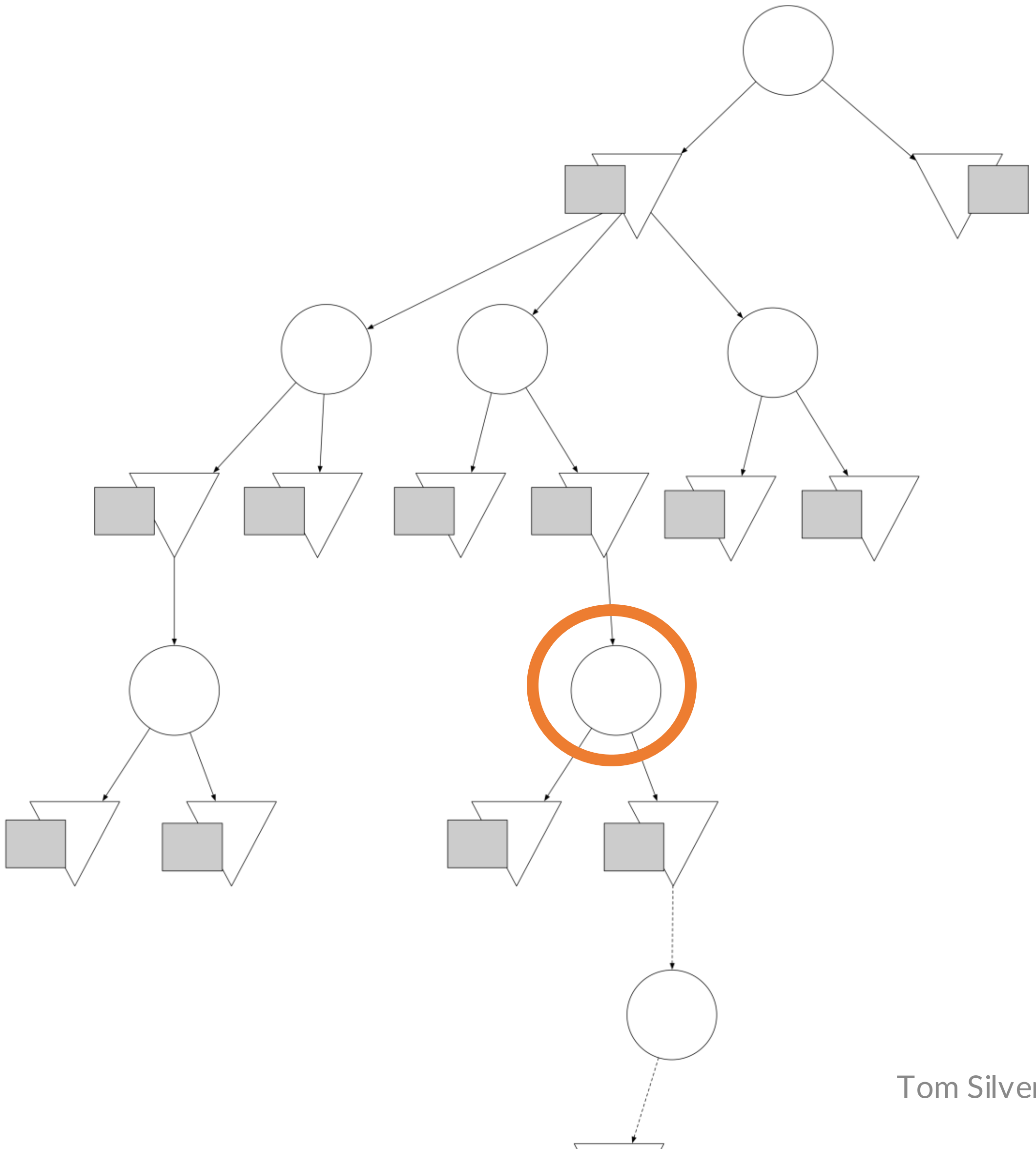
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t}+1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})-1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a})+\mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



SIMULATE$(\texttt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, \texttt{t})$

// Base case: we've never visited this state at this depth before
if $(\texttt{t}, \texttt{s}, \texttt{a}) \notin \texttt{N}$ for an arbitrary $\texttt{a} \in \mathcal{A}$
    for $\texttt{a} \in \mathcal{A}$
        $\texttt{N}(\texttt{t}, \texttt{s}, \texttt{a}) = 0$
        $\texttt{Q}(\texttt{t}, \texttt{s}, \texttt{a}) = 0$
    return $\boxed{\text{ESTIMATEHEURISTIC}(\texttt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)}$ // Run random rollouts
$\texttt{a} = \text{EXPLORE}(\texttt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, \texttt{t})$ // differs between MCTS algs
$\texttt{ns} \sim \texttt{P}(\cdot \mid \texttt{s}, \texttt{a})$
$\texttt{qtsa} = R(\texttt{s}, \texttt{a}, \texttt{ns}) + \gamma\text{SIMULATE}(\texttt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \texttt{Q}, \texttt{N}, \texttt{t} + 1)$
$\texttt{N}(\texttt{t}, \texttt{s}, \texttt{a}) = \texttt{N}(\texttt{t}, \texttt{s}, \texttt{a}) + 1$
$\texttt{Q}(\texttt{t}, \texttt{s}, \texttt{a}) = \frac{(\texttt{N}(\texttt{t},\texttt{s},\texttt{a})-1)\texttt{Q}(\texttt{t},\texttt{s},\texttt{a})+\texttt{qtsa}}{\texttt{N}(\texttt{t},\texttt{s},\texttt{a})}$
return $\texttt{Q}(\texttt{t}, \texttt{s}, \texttt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before

if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$

  for $\mathtt{a} \in \mathcal{A}$

    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

  return $\boxed{\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)}$ // Run random rollouts

$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs

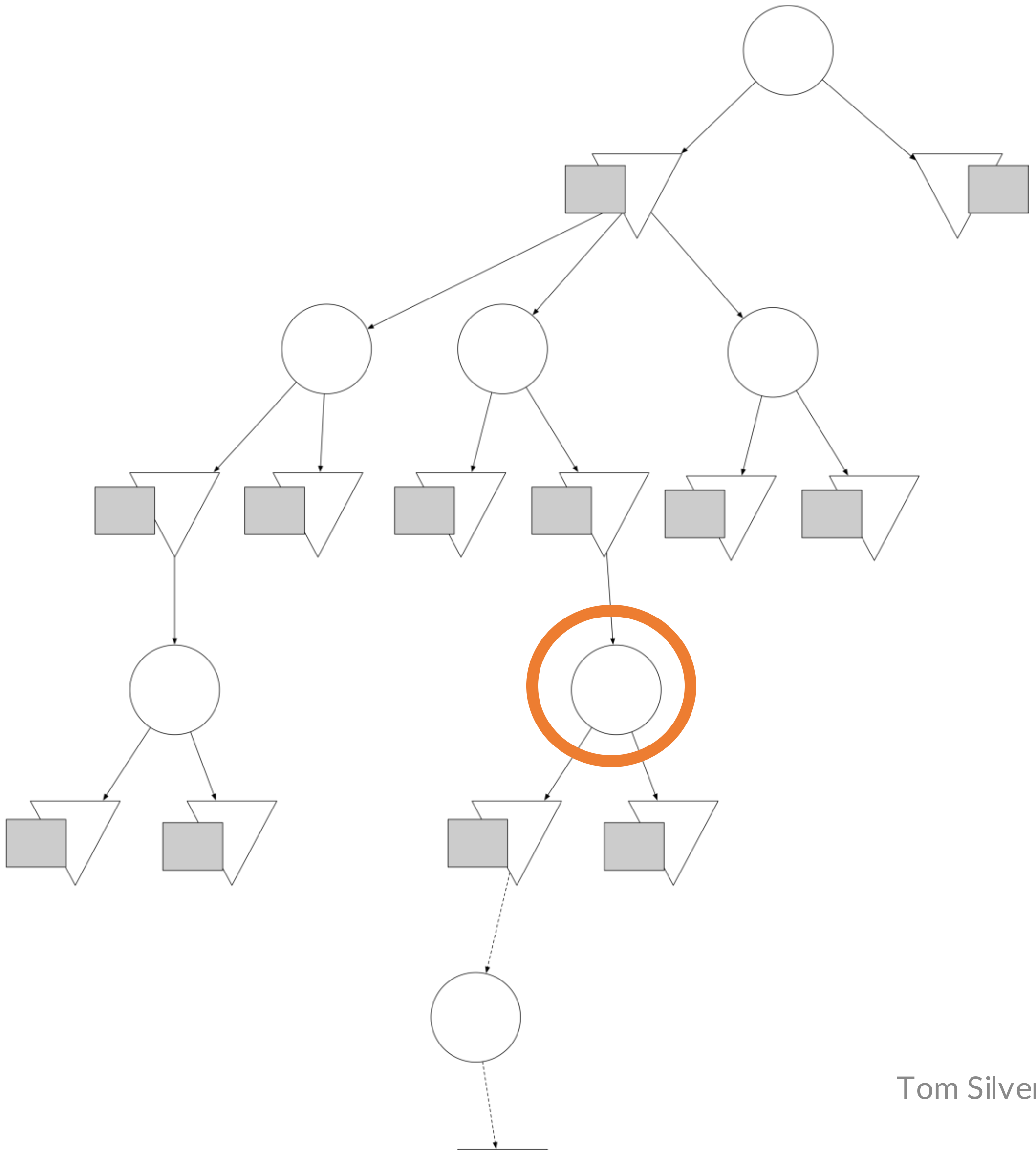
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$

$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$

$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$

$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$

return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before

if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$

    for $\mathtt{a} \in \mathcal{A}$

        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$

    return $\boxed{\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)}$ // Run random rollouts

$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs

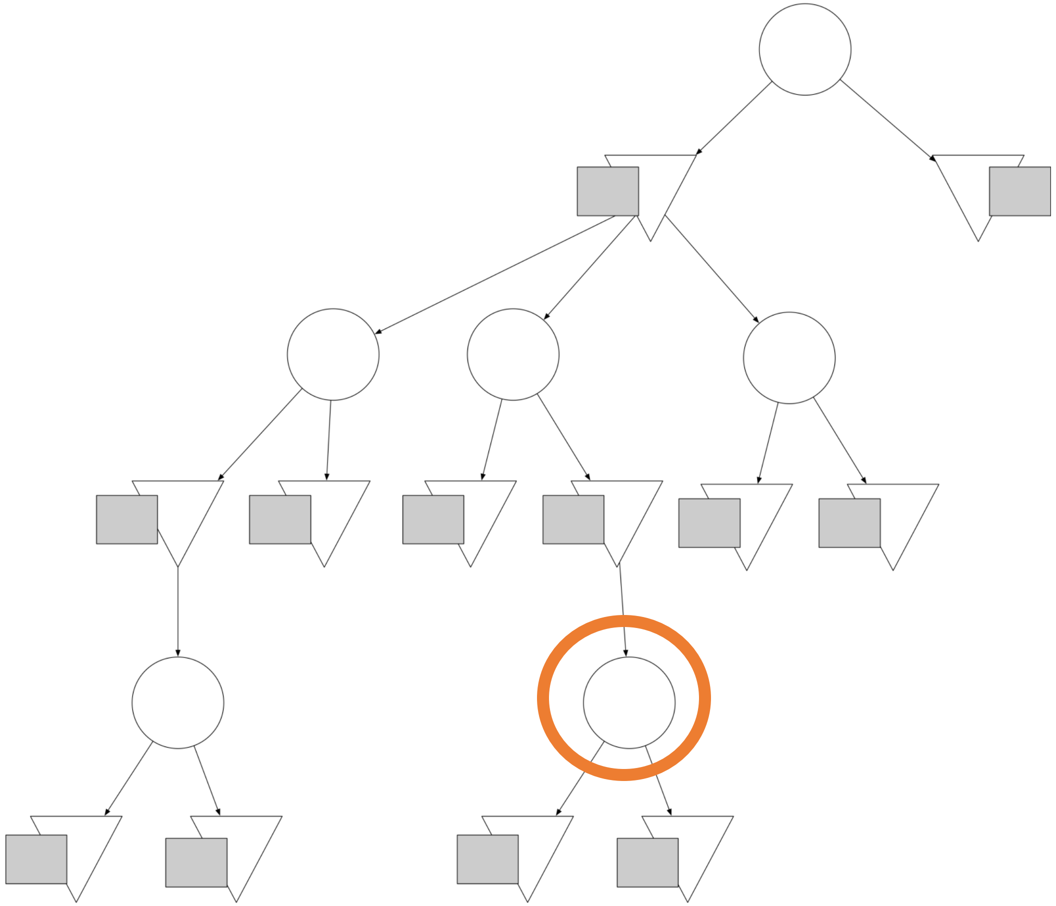
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$

$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$

$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$

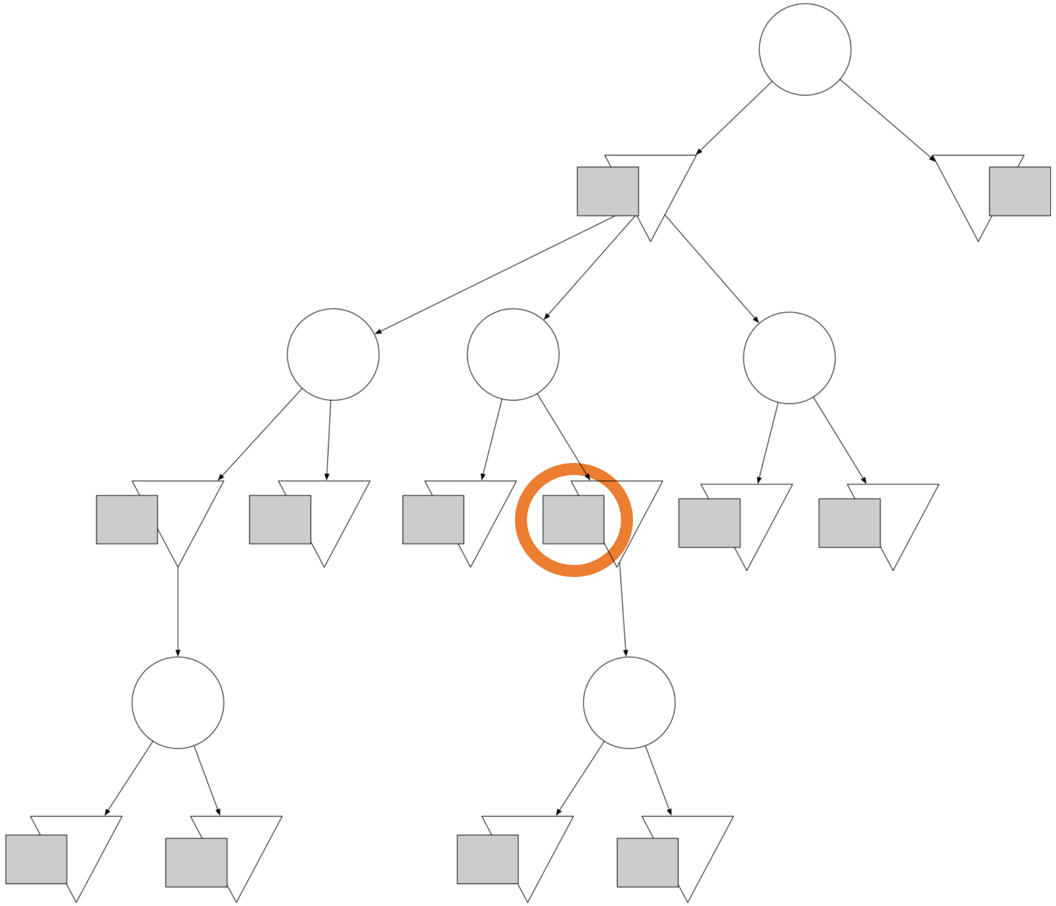
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$

return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



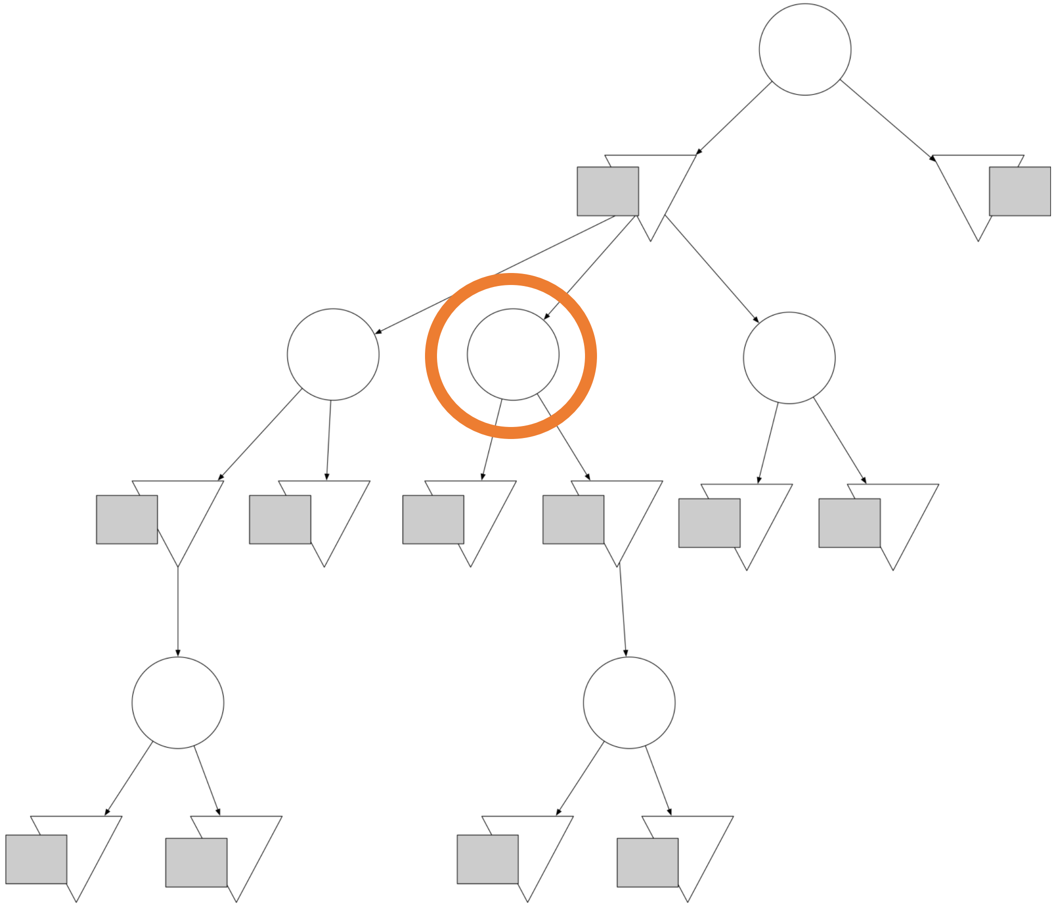
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \boxed{\textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t}+1)}$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})-1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a})+\mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



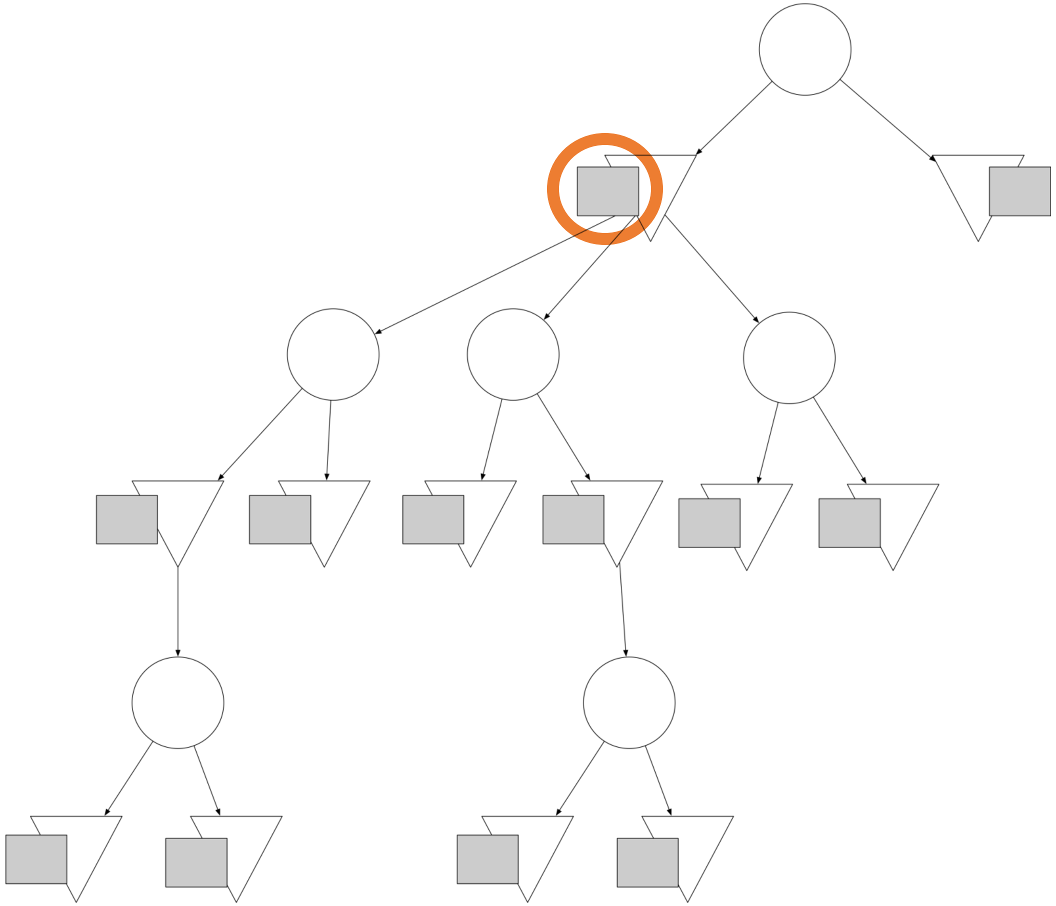
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
    for $\mathtt{a} \in \mathcal{A}$
        $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
        $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



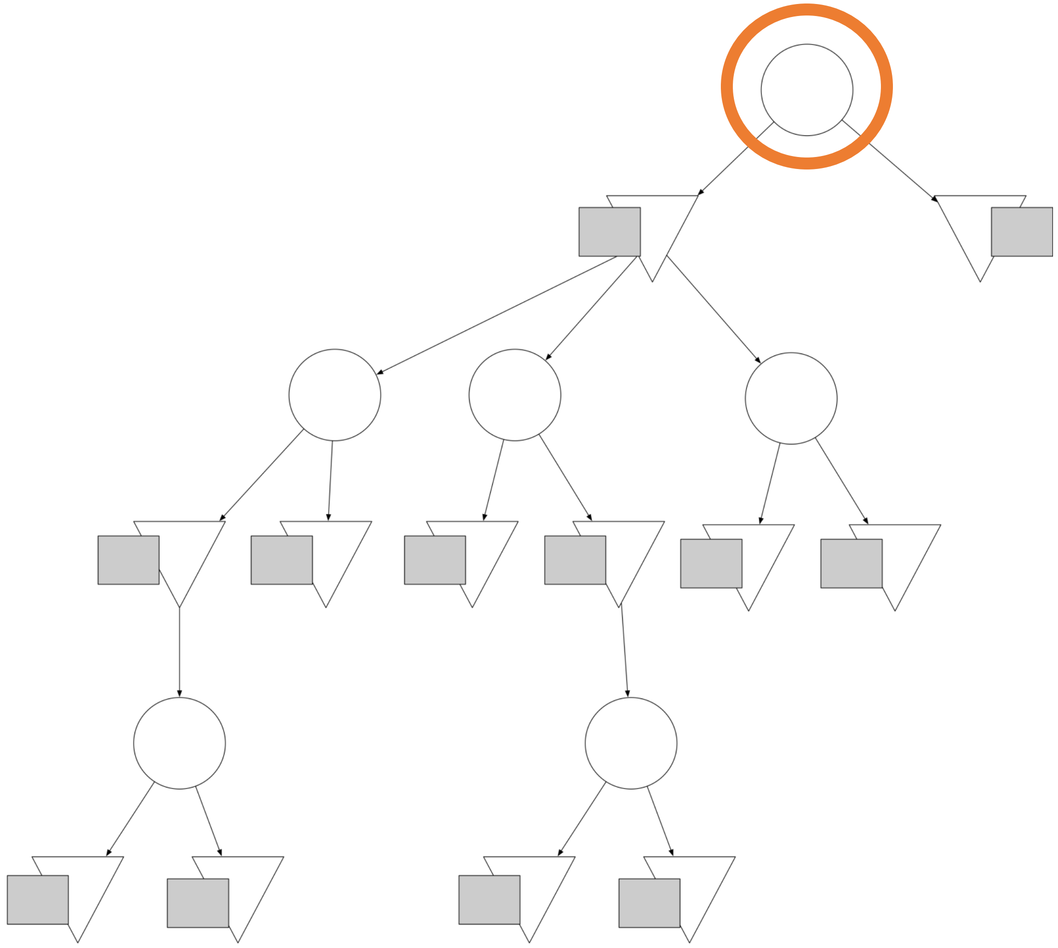
$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \boxed{\textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)}$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$



$\textsc{Simulate}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$

// Base case: we've never visited this state at this depth before
if $(\mathtt{t}, \mathtt{s}, \mathtt{a}) \notin \mathtt{N}$ for an arbitrary $\mathtt{a} \in \mathcal{A}$
  for $\mathtt{a} \in \mathcal{A}$
    $\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
    $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = 0$
  return $\textsc{EstimateHeuristic}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma)$ // Run random rollouts
$\mathtt{a} = \textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t})$ // differs between MCTS algs
$\mathtt{ns} \sim \mathrm{P}(\cdot \mid \mathtt{s}, \mathtt{a})$
$\mathtt{qtsa} = R(\mathtt{s}, \mathtt{a}, \mathtt{ns}) + \gamma \textsc{Simulate}(\mathtt{ns}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N}, \mathtt{t} + 1)$
$\mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \mathtt{N}(\mathtt{t}, \mathtt{s}, \mathtt{a}) + 1$
$\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a}) = \frac{(\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a}) - 1)\mathtt{Q}(\mathtt{t},\mathtt{s},\mathtt{a}) + \mathtt{qtsa}}{\mathtt{N}(\mathtt{t},\mathtt{s},\mathtt{a})}$
return $\mathtt{Q}(\mathtt{t}, \mathtt{s}, \mathtt{a})$

UCT: MCTS + UCB

- Probably the most popular algorithm in the MCTS family is **Upper Confidence Trees (UCT)**.
- UCT uses the exploration bonus from UCB to select actions.

$\textsc{Explore}(\mathtt{s}, \mathcal{S}, \mathcal{A}, P, R, \gamma, \mathtt{Q}, \mathtt{N})$

1. $\mathtt{Ns} = \sum_{a \in \mathcal{A}} \mathtt{N}(\mathtt{s}, \mathtt{a})$
2. // c is the hyperparameter discussed in UCB slide
3. **return** $\operatorname{argmax}_{\mathtt{a} \in \mathcal{A}} \mathtt{Q}(\mathtt{s}, \mathtt{a}) + c\sqrt{\frac{\log \mathtt{Ns}}{\mathtt{N}(\mathtt{s},\mathtt{a})}}$

Summary

- Sparse sampling: expectimax search, but instead of full Bellman backups, use sampling to approximate
- Multi-armed bandits: select actions to minimize regret
- Monte Carlo Tree Search: sparse sampling + MAB exploration techniques + rollouts to estimate heuristics